1.1 Poisson equation $\nabla^2\phi = -4\pi\rho$

$\phi = \frac{q}{r}$ implies $\nabla^2\phi = q\left(-4\pi\delta^3(r)\right)$ hence $\boxed{\rho = q\,\delta^3(r)}$

1.2 $\phi = \frac{q}{r}e^{-\alpha r}$ implies

$$\nabla^2\phi = q\left(-4\pi\delta^3(r)\right) + q\,\nabla^2\,\frac{e^{-\alpha r}-1}{r}$$

$$= q\left(-4\pi\delta^3(r)\right) + q\,\partial_i\left(e^{-\alpha r}\frac{r_i}{r^2}(-\alpha) - \frac{r_i}{r^3}\left(e^{-\alpha r}-1\right)\right)$$

$$= q\left(-4\pi\delta^3(r)\right) + q\left(e^{-\alpha r}(-\alpha)^2\frac{r^2}{r^3} + e^{-\alpha r}\frac{3}{r^2}(-\alpha) - 2e^{-\alpha r}\frac{r^2}{r^4}(-\alpha)\right.$$

$$\left. - \frac{3}{r^3}\left(e^{-\alpha r}-1\right) + 3\frac{r^2}{r^5}\left(e^{-\alpha r}-1\right) - \frac{r^2}{r^4}e^{-\alpha r}(-\alpha)\right)$$

$$= q\left(-4\pi\delta^3(r)\right) + q\left(e^{-\alpha r}\frac{\alpha^2}{r} + \cancel{e^{-\alpha r}\frac{-3\alpha}{r^2}} + \cancel{e^{-\alpha r}\frac{2\alpha}{r^2}}\right.$$

$$\left. - \cancel{\frac{3}{r^3}e^{-\alpha r}} + \cancel{\frac{3}{r^3}} + \cancel{3\frac{e^{-\alpha r}}{r^3}} - \cancel{\frac{3}{r^3}} + \cancel{e^{-\alpha r}\frac{\alpha}{r^2}}\right)$$

$$= q\left(-4\pi\delta^3(r)\right) + q\,\alpha^2\,\frac{e^{-\alpha r}}{r}$$

hence $\boxed{\rho = q\,\delta^3(r) - q\,\frac{\alpha^2}{4\pi}\,\frac{e^{-\alpha r}}{r}}$

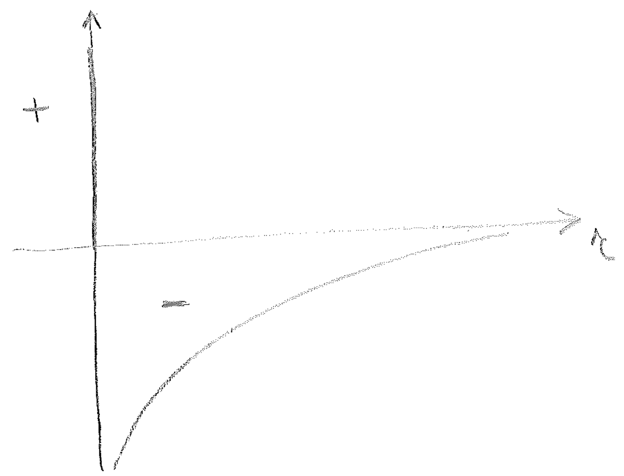
<u>obs</u> in spherical coordinates $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \dots$

hence

$$\nabla^2\,\frac{e^{-\alpha r}-1}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}\,r^2\left(-\frac{1}{r^2}\left(e^{-\alpha r}-1\right) - \alpha\,\frac{e^{-\alpha r}}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(-e^{-\alpha r}+1-\alpha r\,e^{-\alpha r}\right)$$

$$= \frac{1}{r^2}\left(\cancel{\alpha e^{-\alpha r}} - \cancel{\alpha e^{-\alpha r}} + \alpha^2 r\,e^{-\alpha r}\right) = \frac{\alpha^2}{r}e^{-\alpha r}$$

Discussion



this distribution describes a screened pointlike positive charge.

1.3 $$Q = \int d^3r \; \rho = \int d^3r \; \rho\, \delta^3(r) - \rho \frac{\alpha^2}{4\pi} \frac{e^{-\alpha r}}{r}$$

$$= \rho - \rho \frac{\alpha^2}{4\pi} \int d^3r \; \frac{e^{-\alpha r}}{r} = \rho \left(1 - \alpha^2 \int_0^\infty dr \; r\, e^{-\alpha r} \right)$$

$$= \rho \left(1 + \alpha^2 \frac{\partial}{\partial \alpha} \underbrace{\int_0^\infty dr \; e^{-\alpha r}}_{= \frac{1}{-\alpha} e^{-\alpha r} \Big|_0^\infty} \right) = \rho \left(1 + \alpha^2 \underbrace{\frac{\partial}{\partial \alpha} \frac{1}{\alpha}}_{-\frac{1}{\alpha^2}} \right) = 0$$

hence $\boxed{Q = 0}$.

2.1 $\vec{v} = \vec{\omega} \times \vec{r}$

$$\rho(r) = \frac{q}{\frac{4}{3}\pi a^3} = \frac{3}{4\pi}\frac{q}{a^3} \quad \text{for } r \le a \;;\; \rho(r) = 0 \text{ for } r > a$$

$$\vec{j}(\vec{r}) = \rho(r)\,\vec{v} = \begin{cases} \frac{3}{4\pi}\frac{q}{a^3}\,\vec{\omega}\times\vec{r} & \text{for } r \le a \\ 0 & \text{for } r > a \end{cases}$$

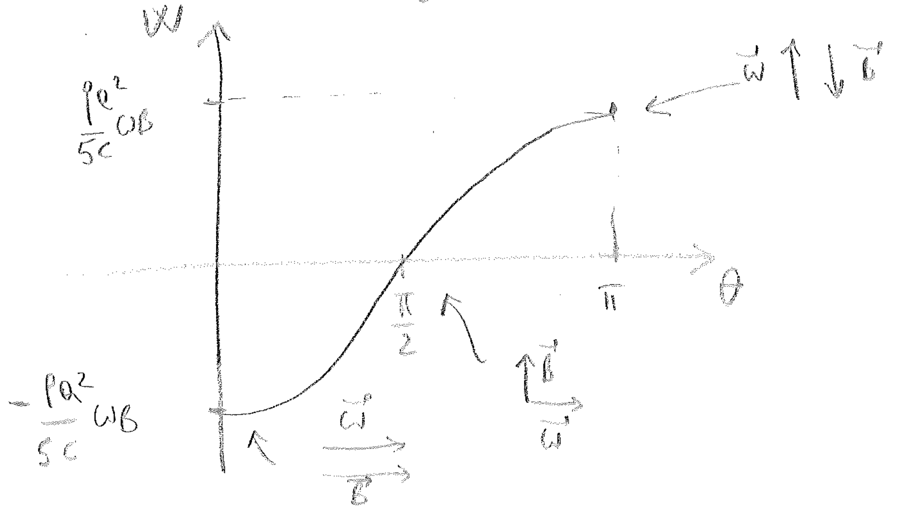
$$\vec{m} = \frac{1}{2c}\int d^3r \;\; \vec{r}\times\vec{j} = \frac{1}{2c}\int_{r\le a} d^3r \; \frac{3}{4\pi}\frac{q}{a^3}\underbrace{\vec{r}\times(\vec{\omega}\times\vec{r})}_{\vec{\omega}\,r^2 - \vec{r}\;\vec{\omega}\cdot\vec{r}}$$

$$= \frac{3}{8\pi}\frac{1}{c}\frac{q}{a^3}\int_{r\le a} d^3r \;\; \vec{\omega}\,r^2 - \vec{\omega}\,\frac{r^2}{3}$$

$$= \frac{3}{8\pi}\frac{1}{c}\frac{q}{a^3}\,\vec{\omega}\,\frac{2}{3}\,4\pi\int_0^a dr\; r^4 = \frac{1}{5c}\,q a^2\,\vec{\omega}$$

$$\boxed{\vec{m} = \frac{q a^2}{5c}\,\vec{\omega}}$$

2.2 $W = -\vec{m}\cdot\vec{B} = -\frac{q a^2}{5c}\,\omega B\cos\theta$



2.3 $v_t = a\omega$; $|\vec{m}| = \frac{ea^2}{5c}|\vec{\omega}|$; $a = \frac{e^2}{mc^2}$

$$\begin{cases} |\vec{m}| = \frac{ea^2}{5c}|\vec{\omega}| = \frac{ea^2}{5c}\frac{v_t}{a} = \frac{e}{5c}\frac{e^2}{mc^2}v_t \\ |\vec{m}| = \frac{e}{2mc^2}\hbar \end{cases}$$

which implies $\frac{e}{2mc^2}\hbar = \frac{e^3}{5mc^3}v_t$

or $$\boxed{\frac{v_t}{c} = \frac{5}{2}\frac{\hbar}{e^2} = \frac{5}{2}\cdot 137 \gg 1}$$

Since $\frac{v_t}{c} \gg 1$, a classical interpretation of the electron's spin is not possible.

3.1 Maxwell equations in vacuum

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = 0 \end{cases}$$

$$\vec{E} = E_0 \begin{pmatrix} \cos(kz-\omega t) \\ \sin(kz-\omega t) \\ 0 \end{pmatrix}$$

if $\vec{E} = \vec{E}(\vec{k}\cdot\vec{r}-\omega t)$ and $\vec{k} = k\vec{e}_z$, then the Maxwell equations may be also written as

$$\begin{cases} \vec{k}\cdot\vec{E}' = 0 \\ \vec{k}\cdot\vec{B}' = 0 \\ \vec{k}\times\vec{E}' - \frac{\omega}{c}\vec{B}' = 0 \\ \vec{k}\times\vec{B}' + \frac{\omega}{c}\vec{E}' = 0 \end{cases}$$

up to a constant field

$$\begin{cases} \vec{k}\cdot\vec{E} = 0 \\ \vec{k}\cdot\vec{B} = 0 \\ \vec{k}\times\vec{E} - \frac{\omega}{c}\vec{B} = 0 \\ \vec{k}\times\vec{B} + \frac{\omega}{c}\vec{E} = 0 \end{cases}$$

with $\omega = kc$, which implies

a solution is: $\vec{B} = \vec{e}_z \times \vec{E}$

$$\boxed{\vec{B} = E_0 \begin{pmatrix} -\sin(kz-\omega t) \\ \cos(kz-\omega t) \\ 0 \end{pmatrix}}$$

3.2 $\vec{F} = e\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$

from $dW = \vec{F}\cdot d\vec{s} = \vec{F}\cdot\vec{v}\,dt$

$$\boxed{\frac{dW}{dt} = \vec{v}\cdot\vec{F} = e\vec{v}\cdot\vec{E}.}$$

3.3 a) $$\frac{d\vec{p}}{dt} = \vec{F} = e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) = e\left(\vec{E} + \frac{\vec{v}}{c}\times(\hat{e}_z\times\vec{E})\right)$$

$$= e\left(\vec{E}\left(1 - \frac{\vec{v}}{c}\cdot\hat{e}_z\right) + \hat{e}_z\,\frac{\vec{v}}{c}\cdot\vec{E}\right)$$

$$\frac{dp_x}{dt} = eE_x\left(1 - \frac{\dot z}{c}\right) = eE_0\left(1 - \frac{\dot z}{c}\right)\cos(kz - \omega t)$$

$$\frac{dp_y}{dt} = eE_y\left(1 - \frac{\dot z}{c}\right) = eE_0\left(1 - \frac{\dot z}{c}\right)\sin(kz - \omega t)$$

$$\frac{dp_z}{dt} = \frac{e}{c}\left(\dot x E_x + \dot y E_y\right) = \frac{e}{c}E_0\left(\dot x\cos(kz-\omega t) + \dot y\sin(kz-\omega t)\right)$$

$$= \frac{\dot W}{c}$$

b) If $\dot W = 0$ then $\frac{dp_z}{dt} = 0$ or $m\ddot z = 0$, which implies

$\dot z = \text{const} \equiv v_{0z}$ and $z(t) = v_{0z}t + z_0$

If $\dot z(t=0) = 0$ then $v_{0z} = 0$: $\boxed{z(t) = z_0}$

From
$$\begin{cases} \frac{dp_x}{dt} = eE_0\cos(kz_0 - \omega t) \\ \frac{dp_y}{dt} = eE_0\sin(kz_0 - \omega t) \end{cases}$$

it follows ($\omega = ck$)

$$m\dot x = p_x = \frac{eE_0}{\omega}\sin(\omega t - kz_0) + \bar p_x$$

$$m\dot y = p_y = \frac{eE_0}{\omega}\cos(\omega t - kz_0) + \bar p_y$$

The condition $\dot W = 0$ implies $p_x\cos(kz_0 - \omega t) + p_y\sin(kz_0 - \omega t) = 0$,

i.e. $$\frac{eE_0}{\omega}\sin(\omega t - kz_0)\cos(\omega t - kz_0) + \bar p_x\cos(\omega t - kz_0) - \frac{eE_0}{\omega}\cos(\omega t - kz_0)\sin(\omega t - kz_0) - \bar p_y\sin(\omega t - kz_0) = 0$$

i.e. $\bar p_x = 0$ and $\bar p_y = 0$

hence

$$m\ddot{x} = \dot{p}_x = \frac{e E_0}{\omega} \sin(\omega t - k z_0)$$

$$m\ddot{y} = \dot{p}_y = \frac{e E_0}{\omega} \cos(\omega t - k z_0)$$

or

$$x = -\frac{e E_0}{m\omega^2} \cos(\omega t - k z_0) + \bar{x}$$

$$y = \frac{e E_0}{m\omega^2} \sin(\omega t - k z_0) + \bar{y}$$

The particle describes in the plane $z = z_0$ a circle centered in $(\bar{x}, \bar{y})$ with radius:

$$\sqrt{(x-\bar{x})^2 + (y-\bar{y})^2} = \sqrt{\frac{e^2 E_0^2}{m^2 \omega^4}} = \frac{e E_0}{m\omega^2}$$

c) Since $\vec{p} = \frac{e E_0}{\omega} \begin{pmatrix} \sin(\omega t - k z_0) \\ \cos(\omega t - k z_0) \\ 0 \end{pmatrix} = \frac{e E_0}{\omega} \begin{pmatrix} -\sin(k z_0 - \omega t) \\ \cos(k z_0 - \omega t) \\ 0 \end{pmatrix}$

it follows that

$$\vec{p} = \frac{e}{\omega} \vec{B}$$

and

$$\vec{p}^2 = \frac{e^2 E_0^2}{\omega^2} \quad \text{or} \quad |\vec{p}| = \frac{e E_0}{\omega}$$