Zübung 9

Many-Body Systems

1.) Fermi degeneracy pressure in stars

- Living stars: fusion of hydrogen into helium releases energy → heats star.

Hot gas has a large pressure that in the case of stars is balanced by gravitational attraction.

Eventually fusion processes end (hydrogen → helium → other elements such as carbon and oxygen) and there is no internal source of energy.

- White dwarf is created
  * Held up against gravitational collapse not by heat, but by electron degeneracy pressure.

a) Carbon → 6 protons, 6 neutrons

Oxygen → 8 protons, 8 neutrons

electrons → contribute very little to mass

$$N_e = N_p = \frac{M/2}{m_p} \Rightarrow \rho_e = \frac{N_e}{V} = \frac{M/2m_p}{\frac{4}{3}\pi R^3}.$$

b.) Given number density ρ_e of degenerate electron gas, calculate Fermi momentum:

$$\rho_e = \frac{K_f^3}{3\pi^2} \Rightarrow K_f^3 V = 3\pi^2 N_e = 3\pi^2 \frac{M}{2m_p}.$$

then total kinetic energy is found by summing over all occupied states:

$$T_e = 2V \int \frac{d^3K}{(2\pi)^3} \cdot \frac{\hbar^2 K^2}{2m_e} = V \frac{\hbar^2}{m_e} \frac{4\pi}{(2\pi)^3} \int_0^{K_f} dK\, K^4$$

(spin-degeneracy)

$$= \frac{\hbar^2}{2\pi^2 m_e} \left(\frac{1}{5} K_f^5\right) V$$

$$= \frac{\hbar^2}{2\pi^2 m_e} \frac{1}{5} K_f^2 \left(3\pi^2 \frac{M}{2m_p}\right)$$

$$= \frac{3\hbar^2}{10\, m_e} \left(\frac{M}{2m_p}\right) \left(3\pi^2 \frac{M}{2m_p} \frac{3}{4\pi R^3}\right)^{2/3}$$

$$= \frac{3\hbar^2}{10 m_e R^2} \left(\frac{M}{2m_p}\right)^{5/3} \left(\frac{9\pi}{4}\right)^{2/3}$$

c.) $$E = T_e + E_g = \frac{3\hbar^2}{10 m_e R^2} \left(\frac{9\pi}{4}\right)^{2/3} \left(\frac{M}{2m_p}\right)^{5/3} - \frac{3GM^2}{5R}.$$

$$\frac{dE}{dR} = \frac{-3\hbar^2}{5 m_e R^3} \left(\frac{9\pi}{4}\right)^{2/3} \left(\frac{M}{2m_p}\right)^{5/3} + \frac{3GM^2}{5R^2} = 0$$

$$\Rightarrow R = \frac{\hbar^2}{GM^2 m_e} \left(\frac{9\pi}{4}\right)^{2/3} \left(\frac{M}{2m_p}\right)^{5/3}$$

$$= \frac{\hbar^2}{G m_e} \left(\frac{81\pi^2}{512 m_p^5}\right)^{1/3} M^{-1/3}.$$

$M = M_\odot \Rightarrow R = 28{,}000$ Km $\quad (R_\odot = 7\times 10^5$ Km$)$

d.) The calculation is exactly as above, except that

$$\frac{M}{2m_p} \rightarrow \frac{M}{m_n} \quad \text{and} \quad m_e \rightarrow m_n$$

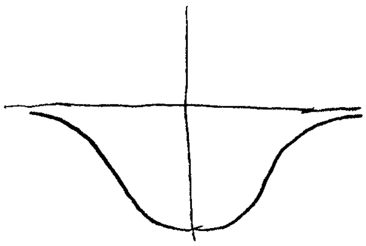
$$\Rightarrow R = \frac{\hbar^2}{G m_n}\left(\frac{81\pi^2}{16\, m_n^5}\right)^{1/3} M^{-1/3}$$

$$M = 1.5\, M_\odot = 1.5\,(2\times 10^{30}\ \mathrm{Kg})$$

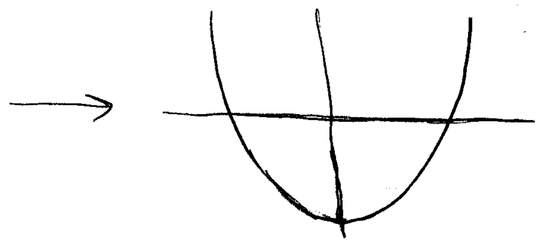
$$\Rightarrow \underline{R = 11\ \mathrm{Km}}$$

Somewhat a coincidence the value agrees so well.

Z10 : Nuclear Shell Model and Collective Excitations



Woods-Saxon potential



Approximate with harmonic oscillator

$$H = \sum_{i=1}^{A} (T_i + U_i) + \frac{1}{2}\sum_{i\neq j} V_{ij}$$

$$U_i = U(r_i) = \frac{1}{2} m\omega^2 r_i^2$$

3D Harmonic Oscillator states :

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)\, Y_l^m(\theta,\phi)$$

$n = 1, 2, 3, \ldots$ number of nodes + 1

$l = 0(s), 1(p), 2(d), \ldots$ orbital angular momentum

$m = -l, -l+1, \ldots, l$ z-projection of orbital ang. mom.

Parity is given by $(-1)^l$.

Energy of eigenstates depends of number N of oscillator quanta : $N = 2(n-1) + l$ and $E_N = (N + \frac{3}{2})\hbar\omega$

States with same N are degenerate.

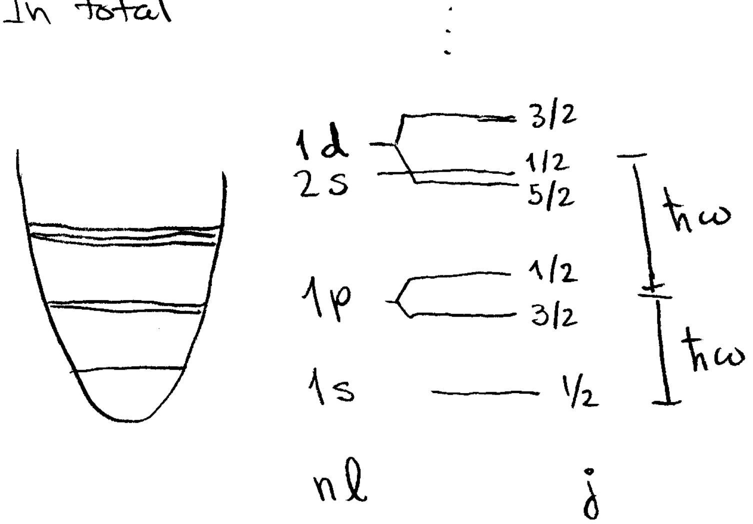
Realistic Woods-Saxon : states with larger l somewhat lower (in general).

Spin degeneracy is removed with spin-orbit interaction :

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left[\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right] \longrightarrow \begin{cases} l/2 & \text{for } j = l + 1/2 \\ -(l+1)/2 & \text{for } j = l - 1/2 \end{cases}$$

$\Rightarrow V_{ls}(r)\,\vec{L}\cdot\vec{S}$ lowers $j = l + 1/2$ state and raises $j = l - 1/2$ for $V_{ls}(r) < 0$.

In total



Degeneracy : $(2j+1)\,2$

↑ protons, neutrons

a) Oscillator constant $\hbar\omega$ related to radius. 3

Degeneracy of N quantum state (N not to be confused with neutron number)

N	n_x	n_y	n_z	degeneracy
0	0	0	0	1
1	1	0	0	
	0	1	0	3
	0	0	1	
2	2	0	0	
	0	2	0	
	0	0	2	
	1	1	0	6
	1	0	1	
	0	1	1	
N	......			$\frac{(N+1)(N+2)}{2}$

Number of nucleons that can fill an N shell:

$$\underset{\text{spin}}{2} \cdot \underset{\text{isospin}}{2} \; \frac{(N+1)(N+2)}{2} = 2(N+1)(N+2)$$

$\Rightarrow$ Total number of nucleons in shells $1, 2, \ldots, N_{max}$

$$A = \sum_{i=0}^{N_{max}} 2(i+1)(i+2) = \sum_{i=0}^{N_{max}} 2i^2 + 6i + 4$$

Recall: $\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$, $\sum_{i=1}^{N} i = \frac{3N(N+1)}{2}$

$$\Rightarrow \quad A \simeq \frac{2}{3} N_{max}^3 \text{ (for large } N\text{)}$$

Now $\langle R^2 \rangle = \frac{1}{A} \sum_{i=1}^{N_{max}} D_i \langle r^2 \rangle_i$

$\uparrow$ degeneracy

To compute $\langle r^2 \rangle_i$, note that $r^2 = x^2 + y^2 + z^2$

and $x = \sqrt{\frac{\hbar}{2m\omega}} \left(a_x + a_x^\dagger \right)$

$$\Rightarrow x^2 = \frac{\hbar}{2m\omega} \left(a_x^2 + a_x^{\dagger 2} + a_x a_x^\dagger + a_x^\dagger a_x \right)$$

Only the last two terms will contribute in an expectation value: $a_x a_x^\dagger |n_x\rangle = (n_x + 1)|n_x\rangle$, $a_x^\dagger a_x |n_x\rangle = n_x |n_x\rangle$, etc.

$$\Rightarrow \langle n_x n_y n_z | r^2 | n_x n_y n_z \rangle$$

$$= \frac{\hbar}{2m\omega} \left(2n_x + 2n_y + 2n_z + 3 \right)$$

$$(N = n_x + n_y + n_z) \Rightarrow \frac{\hbar}{m\omega}\left(N + \frac{3}{2} \right)$$

Therefore, $\langle R^2 \rangle = \frac{1}{A} \sum_{i=1}^{N_{max}} D_i \langle r^2 \rangle_i$

$$= \frac{1}{A} \sum_{i=1}^{N_{max}} \left[2(i+1)(i+2) \right] \left[\frac{\hbar}{m\omega} \left(i + \frac{3}{2} \right) \right]$$

Keeping only the highest power of N_{max} $\left(\sum_{i=1}^{N_{max}} i^3 \simeq \frac{1}{4} N_{max}^4 \right)$

$$\Rightarrow \langle R^2 \rangle = \frac{1}{A} \frac{2\hbar}{m\omega} \frac{1}{4} N_{max}^4$$

$$= \frac{1}{A} \frac{\hbar}{2m\omega} \left(\frac{3}{2} A \right)^{4/3}$$

$$= \frac{\hbar}{2m\omega} \left(\frac{3}{2} \right)^{4/3} A^{1/3}$$

Finally, we know empirically that $R_{nucl} \sim 1.2\, A^{1/3}$ fm,
and for a sphere of constant density

$$\langle R^2 \rangle \doteq \frac{3}{5} R_n^2.$$

Therefore,

$$R_n^2 = \frac{5}{3}\langle R^2\rangle = \frac{5}{3}\frac{\hbar}{2m\omega}\left(\frac{3}{2}\right)^{4/3} A^{1/3} = \left(1.2\, A^{1/3}\right)^2$$

$$\Rightarrow \hbar\omega \doteq \frac{5}{3}\frac{\hbar^2 c^2}{2mc^2}\left(\frac{3}{2}\right)^{4/3}\frac{1}{(1.2)^2} A^{-1/3}$$

$$\simeq 41\, A^{-1/3}\ \text{MeV}.$$

For ^{16}O we have $\hbar\omega = 41\,(16)^{-1/3} \simeq 16$ MeV.

b) Electric multipole transition selection rules:

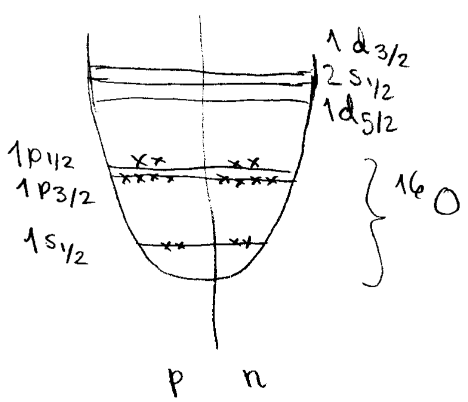
parity change : $\pi_i \pi_f = (-1)^\lambda$

angular momentum : $|J_f - J_i| \leq \lambda \leq J_f + J_i$

For electric dipole transition $\lambda = 1$

$$\Rightarrow \underset{\text{initial } J^\pi}{0^+} \longrightarrow \underset{\text{final } J^\pi}{1^-}$$

c)



Notice that each shell has a definite parity. So, if we consider excitations up to $2\hbar\omega$, only particle-hole excitations between adjacent shells give negative parity.

Only the following particle-hole excitations can couple to $J^\pi = 1^-$:

$$|1p_{3/2}^{-1}\, 1d_{5/2}\rangle \;,\; |1p_{3/2}^{-1}\, 2s_{1/2}\rangle \;,\; |1p_{3/2}^{-1}\, 1d_{3/2}\rangle \;,$$

$$|1p_{1/2}^{-1}, 2s_{1/2}\rangle \;,\; |1p_{1/2}^{-1}, 1d_{3/2}\rangle$$

($|1p_{1/2}^{-1}\, 1d_{5/2}\rangle$ has minimum $J = \frac{5}{2} - \frac{1}{2} = 2$, so not allowed.)

These excitations have energy $\hbar\omega$, compared to $\sim 2\hbar\omega$ of giant dipole resonance. So, giant dipole resonance not a single particle-hole excitation.

c.) The above particle-hole excited states are eigenstates of H_0, but not H. The two-particle interaction V_{ij} mixes these unperturbed states to form new eigenstates of the full Hamiltonian H: $H|\Psi_i\rangle = E_i|\Psi_i\rangle$, and we write

$$|\Psi_j\rangle = \sum_{i=1}^{n} c_j^i |\phi_i\rangle \;,$$ where $|\phi_i\rangle$ is an unperturbed particle-hole excited state satisfying $H|\phi_i\rangle = \varepsilon_i |\phi_i\rangle$.

The eigenstates $|\Psi_i\rangle$ are found by diagonalizing

$$\begin{pmatrix} \varepsilon_1 + V_{11} & V_{12} & \cdots & V_{1n} \\ V_{21} & \varepsilon_2 + V_{22} & \cdots & \\ \vdots & \vdots & \ddots & \\ V_{n1} & & & \varepsilon_n + V_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = E \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} .$$

Assume all $V_{ij} = V_0$ a constant.

then the above eigenvalue equation becomes

$$\begin{pmatrix} \varepsilon_1+V_0 & V_0 & \cdots \\ V_0 & \varepsilon_2+V_0 & \\ \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = E\begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

$$\Rightarrow \quad c_1\varepsilon_1 + V_0(c_1+c_2+\cdots) = E c_1$$
$$c_2\varepsilon_2 + V_0(c_1+c_2+\cdots) = E c_2 \quad \text{, etc.}$$

$$\Rightarrow \quad V_0\sum_{i=1}^{n} c_i = (E-\varepsilon_1)c_1 = (E-\varepsilon_2)c_2 = \cdots$$

$$\Rightarrow \quad c_j = \frac{V_0}{E-\varepsilon_j}\sum_{i}^{n} c_i$$

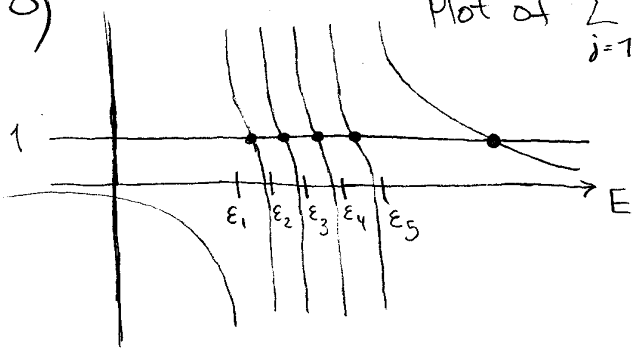
Sum both sides over possible c_j

$$\sum_{j=1}^{n} c_j = \sum_{j=1}^{n}\frac{V_0}{E-\varepsilon_j}\sum_{i=1}^{n} c_i$$

$$\Rightarrow \quad 1 = \sum_{j=1}^{n}\frac{V_0}{E-\varepsilon_j} .$$

Graphical solution

$(V_0 > 0)$



Plot of $\sum_{j=1}^{n}\frac{V_0}{E-\varepsilon_j}$ which has poles at $E=\varepsilon_i$.

Unperturbed eigenvalues $\varepsilon_1 \cdots \varepsilon_5$. Eigenvalues of full Hamiltonian represented by circles. Four eigenvalues squeezed between $\varepsilon_1 \cdots \varepsilon_5$, and fifth eigenvalue lies much above the others.

$\Rightarrow E_i \simeq \varepsilon_i$ for $i = 1, \dots, n-1$.

To estimate E_n, notice that all ε_i are nearly degenerate, so $E_n - \varepsilon_i \approx E_n - \varepsilon_0$ for all i.

$$1 = \sum_{i=1}^{n} \frac{V_0}{E_n - \varepsilon_i} \approx \frac{nV_0}{E_n - \varepsilon_0}$$

$$\Rightarrow \quad E_n \simeq \varepsilon_0 + nV_0 .$$

d) To find the wavefunction $|\Psi_n\rangle$, notice that

$$c_i^n = \underbrace{\frac{V_0}{E_n - \varepsilon_i}}_{\text{nearly constant}} \underbrace{\sum_{j=1}^{n} c_j^n}_{\text{constant}}$$

$$\Rightarrow c_i^n \simeq \frac{1}{\sqrt{n}} \quad \text{(normalized)}$$

$$\Rightarrow |\Psi_n\rangle = \frac{1}{\sqrt{n}} \sum_j |\psi_j\rangle$$

Example: the ^{16}O giant dipole resonance would have wavefunction

$$|\Psi_{GDS}\ J^\pi = 1^-\rangle \simeq \frac{1}{\sqrt{5}} \Big\{ |1p_{3/2}^{-1}\, 1d_{5/2}\rangle + |1p_{3/2}^{-1}\, 2s_{1/2}\rangle$$
$$+ |1p_{3/2}^{-1}\, 1d_{3/2}\rangle + |1p_{1/2}^{-1}\, 2s_{1/2}\rangle + |1p_{1/2}^{-1}\, 1d_{3/2}\rangle \Big\} .$$