

ZENTRALÜBUNG ZUR QUANTENMECHANIK II

WS 2009/10

9. Übungsblatt

(J. Holt, M. Christoforetti, B. Klein, W. Weise)

Besprechung am 18.12.2009

Aufgabe Z9: Fermi Degeneracy Pressure in Stars

Living stars are supported against gravitational collapse by the heat generated through fusion reactions. However, when a star is no longer able to produce sufficient heat through fusion, it may evolve into a type of star known as a white dwarf. White dwarves are held up against gravitational collapse by their electron Fermi degeneracy pressure. Consider a nonrotating, spherical white dwarf of mass M , radius R , and constant density.

- Assuming that the white dwarf is electrically neutral and composed primarily of carbon and oxygen nuclei, estimate the number density $\rho_e = N_e/V$ of electrons in the white dwarf.
- For temperature $T = 0$, show that the (nonrelativistic) total kinetic energy T_e of the degenerate Fermi gas of electrons in the star is given by

$$T_e = \frac{3\hbar^2}{10m_e R^2} \left(\frac{9\pi}{4}\right)^{2/3} \left(\frac{M}{2m_p}\right)^{5/3}.$$

- The gravitational potential energy of a nonrotating star with uniform density is known to be

$$E_g = -\frac{3GM^2}{5R}.$$

Use this to show that the radius of a white dwarf depends on its mass through the relation

$$R = \frac{\hbar^2}{Gm_e} \left(\frac{81\pi^2}{512m_p^5}\right)^{1/3} M^{-1/3},$$

which implies that the volume of the white dwarf is inversely proportional to its mass.

- If the mass of a star is large enough ($M \gtrsim 1.4M_\odot$, where M_\odot is the mass of the sun), a proper relativistic treatment shows that the electron degeneracy pressure will not be sufficient to prevent gravitational collapse, in which case the star can explode as a supernovae and may leave behind a neutron star. Assuming that the neutron star remnant is composed only of neutrons and supported against further gravitational collapse through neutron degeneracy pressure, estimate the radius of a neutron star (assuming constant density) with mass $M = 1.5M_\odot$. Compare this to the typical observational value of $R \sim 10$ km.

Aufgabe Z10: Nuclear Shell Model and Collective Excitations

In the nuclear shell model, one assumes that a nucleon experiences an average potential U due to all other nucleons as well as a small residual pairwise interaction V_{ij} . The Hamiltonian is therefore given by

$$H = \underbrace{\sum_{i=1}^A (T_i + U_i)}_{H_0} + \underbrace{\frac{1}{2} \sum_{i \neq j} V_{ij}}_{H_1},$$

where T_i is the kinetic energy operator and A is the total number of nucleons. We neglect V_{ij} for the time being. In principle, U is given by a Hartree-Fock potential, but in practice a harmonic oscillator potential is more convenient: $U_i = U(r_i) = \frac{1}{2}m\omega^2 r_i^2$.

- a) Empirically, the nuclear radius $R \simeq 1.2A^{1/3}$ fm. Show that in the limit of large A , with $Z = N = A/2$, and assuming a constant nuclear density, the harmonic oscillator constant is given in terms of A by

$$\hbar\omega = 41A^{-1/3} \text{ MeV}.$$

Estimate the energy spacing between adjacent harmonic oscillator shells in ^{16}O .

- b) The cross section for low-energy photoabsorption by nuclei exhibits an electric dipole resonance that is experimentally found to lie at an energy $\hbar\omega \simeq 80A^{-1/3}$ MeV and to have a width of a few MeV. For an even-even nucleus with ground state spin and parity $J^\pi = 0^+$, what are the spin and parity J^π of this “giant dipole resonance” state?
- c) Denote a particle-hole state by $|\psi_i\rangle = |\phi_{j_1}^{-1}\phi_{j_2}\rangle$, for $i = 1, 2, \dots, n$, where $H_0|\psi_i\rangle = \epsilon_i|\psi_i\rangle$ and n is the total number of particle-hole configurations with a given J^π . Consider ^{16}O and list all possible particle-hole excitations that have the same quantum numbers as the giant dipole resonance state (consider excitations up to $2\hbar\omega$). How does the energy of a single particle-hole excitation compare to the energy of the giant dipole resonance?
- c) Now consider the residual two-body interaction V_{ij} which can mix the eigenstates of H_0 . The full Hamiltonian satisfies the Schrödinger equation

$$H|\Psi_i\rangle = E_i|\Psi_i\rangle$$

and we can write

$$|\Psi_j\rangle = \sum_{i=1}^n c_j^i |\psi_i\rangle.$$

Consider a schematic model in which the matrix elements $\langle\psi_i|V|\psi_j\rangle = V_0$, a constant for all i, j . Show that for $V_0 > 0$, $E_i \simeq \epsilon_i$ for $i = 1, 2, \dots, n-1$ and that $E_n \simeq \epsilon_n + nV_0$.

- d) Show that this high-lying state $|\Psi_n\rangle$ is a coherent superposition of all allowed particle-hole states. We therefore interpret the giant dipole resonance as a collective state consisting of the superposition of many particle-hole states.