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Nuclear Physics A 760 (2005) 110–138
NUCLEAR PHYSICS A

Naïve dimensional analysis for three-body forces without pions

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Received 7 March 2005; received in revised form 13 May 2005; accepted 31 May 2005

Available online 29 June 2005

Abstract

For systems of three identical particles in which short-range forces produce shallow two-particle bound states, and in particular for the "pionless" effective field theory of nuclear physics, I extend and systematise the power-counting of three-body forces to all partial waves and orders, including external currents. With low-energy observables independent of the details of short-distance dynamics, the typical strength of a three-body force is determined from the superficial degree of divergence of the three-body diagrams which contain only two-body forces. This naïve dimensional analysis must be amended as the asymptotic solution to the leading-order Faddeev equation depends for large off-shell momenta crucially on the partial wave and spin combination of the system. It is shown by analytic construction to be weaker than expected in most channels with angular momentum smaller than 3. This demotes many three-nucleon forces to high orders. Observables like the ${}^4S_{3/2}$-scattering length are less sensitive to three-nucleon forces than guessed. I also comment on the Efimov effect and limit-cycle for non-zero angular momentum.

PACS: 02.30.Rz; 02.30.Uu; 11.80.Jy; 13.75.Cs; 14.20.Dh; 21.30.-x; 25.40.Dn; 27.10.+h

Keywords: Effective field theory; Three-body system; Three-body force; Faddeev equation; Partial waves; Mellin transform

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0375-9474/$ – see front matter
doi:10.1016/j.nuclphysa.2005.05.202

1. Introduction

Three-body forces parameterise the interactions between three particles on scales much smaller than what can be resolved by two-body interactions. Traditionally, they were often introduced a posteriori to cure discrepancies between experiment and theory, but such an approach is, of course, untenable when data are scarce or absent, predictive power is required, or one- or two-body properties are extracted from three-body data.

The pivotal promise of an Effective Field Theory (EFT) is that it describes all physics below a certain "breakdown scale" to a given accuracy with the minimal set of parameters, and that it hence predicts in a model-independent way the typical strength with which also three-body forces enter in observables. This promise is based on a dimensionless, small parameter: the typical momentum of the low-energy process in units of the breakdown scale, namely, of the scale on which details of the short-range interactions are resolved. It allows one to truncate the momentum expansion of the forces at a given level of accuracy, keeping only and all the terms up to a given order, and thus establishes a "power-counting" of all forces. One can then estimate a priori the experimental accuracy necessary to disentangle particular effects like a three-body force in observables.

The power-counting is to a high degree determined by naïve dimensional analysis [1]. As low-energy observables must be insensitive to the details of short-distance physics, they are, in particular, independent of a cut-off Λ employed to regulate the theory at short distances. Typically, a divergence from loop integrations must therefore be cancelled by at least one coefficient $C(\Lambda)$ in the EFT Lagrangean, which thus also encodes short-distance dynamics. This counter-term enters hence at the same order as the first divergence which it must absorb. It ensures that the EFT is cut-off independent at each order, and therefore renormalisable and self-consistent. With the running of the counter-term thus determined, its initial condition provides an unknown, free parameter which has to be found from experiment. Naïve dimensional analysis assumes now that the typical size of the counter-term is "natural", i.e., at most of the same magnitude as the size of its running: $C(\Lambda) \sim C(2\Lambda) \sim C(2\Lambda) - C(\Lambda)$. Thus, it guarantees that the EFT contains at a given order the minimal number of free parameters which are necessary to render the theory renormalisable, and by that also the minimal number of independent low-energy coefficients necessary to describe all low-energy phenomenology to a given level of accuracy.

When all interactions are treated perturbatively, like in chiral perturbation theory in the purely mesonic sector, naïve dimensional analysis amounts to little more than counting the mass-dimension of an interaction [1]. I will demonstrate that such reasoning becomes, however, too simplistic when some interactions in the EFT must be treated non-perturbatively. This is the case in nuclear physics, and for some systems of atomic physics.

While the separation of scales between low-energy and high-energy degrees of freedom in nuclear physics makes it an ideal playground for EFT-methods, finding such a power-counting proves a difficult goal for few-nucleon systems. To establish a formalism which is self-consistent, agrees with nuclear phenomenology and can firmly be rooted in QCD, one has to cope with shallow real and virtual few-nucleon bound-states. The deuteron size of ≈ 4.3 fm, for example, seems not to be connected even to the soft scales of QCD, e.g., the pion mass or decay constant. The effective low-energy degrees of freedom and symmetries of QCD dictate a unique Lagrangean; but there are conceptually quite different ordering-schemes available for the few-nucleon system which lead to different, experimentally falsifiable predictions. The naïvest versions perturb around the free theory and hence cannot accommodate shallow bound-states at all. They are self-consistent, but not consistent with nature. Weinberg [2] proposed to build few-nucleon systems from a nucleon–nucleon potential which consists of contact interactions and pion-exchanges constrained by chiral symmetry. The interactions in the potential are ordered following naïve dimensional analysis as if the theory would be perturbative, and the potential is then iterated to produce the unnaturally large length scales in the two-nucleon system by fine-tuning between long- and short-distance contributions. Few-nucleon interactions are added using the same prescription. Whether this approach correctly and self-consistently reproduces QCD at low energies in the three and more nucleon sector is an open question.

However, low-lying few-body bound-states also offer the opportunity for a more radical approach: for momenta below the pion mass, the only forces can be taken to be point-like two and more nucleon interactions. This nuclear effective field theory with pions integrated out (EFT($\not\pi$)) is in the two-nucleon system manifestly self-consistent and proves—on quite general grounds—to be the correct version of QCD at extremely low energies, once fine-tuning is observed, see Refs. [3–5] for recent reviews. A plethora of pivotal physical processes which are both interesting in their own right and important for astrophysical applications and fundamental questions, e.g., big-bang nucleosynthesis and static neutron properties, were investigated with high accuracy. One obtains usually quite simple, analytic results, and most of the coefficients are determined by simple, well-known long-range observables. Recently, a manifestly self-consistent power-counting for the three-nucleon forces of EFT($\not\pi$) in the $^2S_{1/2}$-wave of Nd scattering was established [6–8]. First high-accuracy calculations also including external sources are now performed [9]. A remarkable phenomenon of this channel is that the first three-body force appears already at leading order to stabilise the wave-function against collapse [10], leading to a new renormalisation group phenomenon, the "limit-cycle" [11–13], manifested also by the Efimov effect [11,14]. This can also be shown using a subtraction method [15]. It was also confirmed by an analysis of the renormalisation group flow in the position-space version of the problem [16].

EFT($\not\pi$) is universal in a dual sense. First, its methods can be applied to a host of systems in which short-range forces conspire to produce shallow two-particle bound states: one example are identical spinless bosons, found in bound-states of neutral atoms like the ^{4}He dimer and trimer which are bound by van der Waals forces, or loss rates in Bose–Einstein condensates near Feshbach resonances, see Ref. [17] for a review. Our results are thus readily taken over to such systems. Second, any consistent EFT of nucleons and pions must reduce to EFT($\not\pi$) in the extreme low-energy limit. Therefore, lessons learned in the latter shed light on the consistent systematisation of the former. As EFTs are model-independent, considerably more sophisticated and computationally involved potential-model calculations must agree with the predictions of EFT($\not\pi$) when they reproduce the two- and three-body data which are used as input for EFT($\not\pi$) to the same level of accuracy.

This article confronts the power-counting of three-body forces in any three-particle system with large two-particle scattering lengths and only contact interactions. It is organised as follows. In Section 2.1, the necessary foundations are summarised. After establishing the superficial degree of divergence of diagrams which contain only two-body forces in Section 2.2, the far off-shell amplitude of the leading-order Faddeev equation in each partial wave is determined analytically in Section 2.4 together with a short discussion of the Efimov effect in non-integer partial waves. Section 2.4 then classifies at which order a given three-body force is needed to render cut-off independent results. Physically relevant consequences are discussed in Section 3, together with some caveats. After the conclusions (Section 4), Appendix A sketches some mathematical details. I also correct some errors in a brief summary of some of the results in Ref. [18].

2. Three-body forces in EFT($\not\pi$)

2.1. Three-body systems with large two-body scattering length

We consider three identical particles N of mass M interacting only with contact forces such that two particles form a shallow real or virtual two-body bound-state d. As the steps leading to the leading-order (LO) scattering amplitude $dN \to dN$ were often described in the literature, they are not covered here; see Ref. [19] also for the notation used in the following. For convenience, a "deuteron" field is introduced as the auxiliary field which describes scattering between two particles with an anomalously large scattering length $1/\gamma$ [20–23]. Its propagator is therefore given by the LO-truncation of the effective-range expansion [24]:

$$D(q_0,\vec{q}) = \frac{1}{\gamma - \sqrt{\frac{\vec{q}^2}{4} - Mq_0 - \mathrm{i}\epsilon}}. \tag{2.1}$$

A real bound-state d has at this order the binding energy $\gamma^2/M \ll \Lambda_{\not\pi}$, much smaller than the breakdown scale of EFT on which new degrees of freedom are resolved. In the three-body system, an infinite number of diagrams contributes at LO, see Fig. 1. The corresponding Faddeev equation for scattering between the auxiliary field d and the remaining particle, first derived by Skorniakov and Ter-Martirosian [25], is unitarily equivalent to the original problem of scattering between three particles [23,26]. One finds for Nd scattering in the lth partial wave in the centre-of-mass system the integral equation for the half off-shell amplitude (before wave-function renormalisation)

$$\begin{aligned} t_\lambda^{(l)}(E;k,p) &= 8\pi\lambda\mathcal{K}^{(l)}(E;k,p) \\ &\quad - \frac{4}{\pi}\lambda\int_0^\infty \mathrm{d}q\, q^2\mathcal{K}^{(l)}(E;q,p)\,D\left(E-\frac{q^2}{2M},q\right)t_\lambda^{(l)}(E;k,q). \end{aligned} \tag{2.2}$$

With the kinematics defined in Fig. 1, $E := \frac{3\vec{k}^2}{4M} - \frac{\gamma^2}{M} - \mathrm{i}\epsilon$ is the total non-relativistic energy; $\vec{k}$ is the relative momentum of the incoming deuteron; $\vec{p}$ is the off-shell momentum of the outgoing one, with $p = k$ the on-shell point; and the projected propagator of the exchanged particle on angular momentum l is

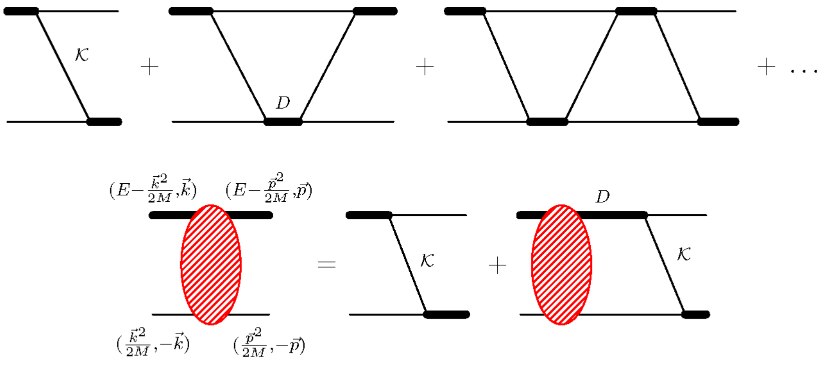

Fig. 1. Resummation of the infinite number of LO three-body diagrams (top) into the corresponding Faddeev integral equation (bottom). Thick line (D): two-nucleon propagator; thin line ($\mathcal{K}$): propagator of the exchanged nucleon; ellipse: LO half off-shell amplitude.

$$\mathcal{K}^{(l)}(E;q,p) := \frac{1}{2}\int_{-1}^{1} \mathrm{d}x \, \frac{P_l(x)}{p^2+q^2-ME+pqx} = \frac{(-1)^l}{pq} Q_l\left(\frac{p^2+q^2-ME}{pq}\right). \tag{2.3}$$

The lth Legendre polynomial of the second kind with complex argument is defined as in [27]

$$Q_l(z) = \frac{1}{2}\int_{-1}^{1} \mathrm{d}t \, \frac{P_l(t)}{z-t}. \tag{2.4}$$

The “spin-parameter” λ depends on the spins of the three particles and how they combine. The values for the physically most relevant three-body systems are summarised in Table 1.

In EFT($\not{\pi}$), Nd scattering in the $S = 1/2$ channel is at first sight described by a more complex integral equation because two-nucleon scattering has two anomalously large scattering lengths: $1/\gamma_s$ in the $^1\mathrm{S}_0$-channel, and $1/\gamma_t$ in the $^3\mathrm{S}_1$-channel. Therefore, two cluster-configurations exist in the three-nucleon system: in one, the spin-triplet auxiliary field d_t (the deuteron) combines with the “spectator” nucleon N to total spin $S = 3/2$ (quartet channel) or $S = 1/2$ (doublet channel), depending on whether the deuteron and nucleon spins are parallel or antiparallel. In the other, the spin-singlet auxiliary field d_s combines with the remaining nucleon to total spin $S = 1/2$. In the doublet channel, the

Table 1
The spin-parameter λ for physical systems of identical particles described by (2.2)

$\lambda = 1$	$\lambda = -\frac{1}{2}$
3 spinless bosons	3 nucleons coupled to $S = 3/2$
Wigner-symmetric part of 3 nucleons coupled to $S = 1/2$	Wigner-antisymmetric part of 3 nucleons coupled to $S = 1/2$

Faddeev equation is thus two-dimensional: the amplitude $t^{(l)}_{d,tt}$ stands for the $Nd_t \to Nd_t$-process, $t^{(l)}_{d,ts}$ for the $Nd_t \to Nd_s$-process, and with $D_{s/t}$ defined analogously to (2.1):

$$\begin{aligned}
&\begin{pmatrix} t^{(l)}_{d,tt} \\ t^{(l)}_{d,ts} \end{pmatrix}(E;k,p) \\
&\quad = 2\pi\mathcal{K}^{(l)}(E;k,p)\begin{pmatrix} 1 \\ -3 \end{pmatrix} \\
&\qquad - \frac{1}{\pi}\int_0^\infty \mathrm{d}q\, q^2 \mathcal{K}^{(l)}(E;q,p)\begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \\
&\qquad \times \begin{pmatrix} D_t\big(E-\frac{q^2}{2M},p\big) & 0 \\ 0 & D_s\big(E-\frac{q^2}{2M},p\big) \end{pmatrix}\begin{pmatrix} t^{(l)}_{d,tt} \\ t^{(l)}_{d,ts} \end{pmatrix}(E;k,q). \qquad (2.5)
\end{aligned}$$

In the following, we are only interested in the unphysical short-distance behaviour of the amplitudes, i.e., in the UV-limit for the half off-shell momenta of (2.5): $p,q \gg k,E,\gamma_{s/t}$. This suffices to determine in Section 2.4 the order at which divergences need to be cancelled by counter-terms parameterising three-nucleon interactions. In this limit, the NN scattering amplitudes are automatically Wigner-$SU(4)$-symmetric, i.e., symmetric under arbitrary combined rotations of spin and isospin [6,28,29]:

$$\lim_{q\gg E,\gamma_{s/t}} D_{s/t}\left(E-\frac{\vec{q}^{\,2}}{2M},\vec{q}\right) = \lim_{q\gg E,\gamma} D\left(E-\frac{\vec{q}^{\,2}}{2M},\vec{q}\right) = -\frac{2}{\sqrt{3}}\frac{1}{q}. \qquad (2.6)$$

Building the following linear combinations which are symmetric, respectively, antisymmetric under Wigner transformations,

$$t^{(l)}_{\mathrm{Ws}} := \frac{1}{2}\big(t^{(l)}_{d,tt} - t^{(l)}_{d,ts}\big), \qquad t^{(l)}_{\mathrm{Wa}} := \frac{1}{2}\big(t^{(l)}_{d,tt} + t^{(l)}_{d,ts}\big), \qquad (2.7)$$

decouples thus the Faddeev equations of the doublet channel [6]:

$$\begin{aligned}
\begin{pmatrix} t^{(l)}_{\mathrm{Ws}} \\ t^{(l)}_{\mathrm{Wa}} \end{pmatrix}(p) &= 4\pi\mathcal{K}^{(l)}(0;0,p)\begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \\
&\quad + \frac{4}{\sqrt{3}\pi}\int_0^\infty \mathrm{d}q\, q^2\mathcal{K}^{(l)}(0;q,p)\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{q}\begin{pmatrix} t^{(l)}_{\mathrm{Ws}} \\ t^{(l)}_{\mathrm{Wa}} \end{pmatrix}(q). \qquad (2.8)
\end{aligned}$$

$t^{(l)}_{\mathrm{Wa}}$ obeys in this limit the same integral equation as the quartet-channels, and $t^{(l)}_{\mathrm{Ws}}$ is identical to the one for three spinless bosons [6,10]. The problem to construct the UV-behaviour of the three-body system with large two-body scattering length simplifies hence to constructing the solution of just one integral equation:

$$t^{(l)}_\lambda(p) = 8\pi\lambda \lim_{k\to 0}\mathcal{K}^{(l)}\left(\frac{3k^2}{4}-\gamma^2;k,p\right) + \frac{8\lambda}{\sqrt{3}\pi}(-1)^l\int_0^\infty \frac{\mathrm{d}q}{p}\,Q_l\left(\frac{p}{q}+\frac{q}{p}\right)t^{(l)}_\lambda(q), \qquad (2.9)$$

where the amplitude $t_\lambda^{(l)}(p)$ depends only on the off-shell momentum p, the partial wave l, and the spin–isospin combination λ. The normalisation of the inhomogeneous part only provides the overall scale of the solution and is hence irrelevant for the following.

In a slight abuse of terminology, the names "deuteron" and "nucleon" are used in the remainder also when three identical bosons are considered.

2.2. Divergences and three-body forces at higher order

Naïve dimensional analysis is based on the UV-behaviour of the scattering amplitude: as outlined in the introduction, a three-nucleon force is needed at some order as counter-term to absorb cut-off dependence induced in the physical amplitudes by divergences, as the ingredients of the Faddeev equation are refined to include higher-order effects. The running of this three-body force with the cut-off is then assumed to be of the same size as its initial condition, which in turn must be determined from a three-body datum. Equivalently, a three-body datum is needed at the same order as the first divergence which must be absorbed by a three-nucleon interaction. We therefore discuss now the superficial degree of divergence of higher-order corrections steming form the "two-body sector" of the theory.

As will be shown in the next subsection, the half off-shell amplitude at large off-shell momenta $p \gg k$ is asymptotically given by

$$t_\lambda^{(l)}(p) \propto k^l\, p^{-s_l(\lambda)-1}, \tag{2.10}$$

where $s_l(\lambda)$ is in general a complex number which depends on the partial wave l and channel λ. Higher-order corrections to three-nucleon scattering can be obtained by perturbing around the LO solution, see Fig. 2. This is numerically tricky [7,8] also because from next-to-next-to-leading order (N^2LO) on, the full LO off-shell amplitude must be computed and convoluted numerically with the corrections, see the centre bottom graph in Fig. 2. However, it allows a simple determination of the order at which the first divergence occurs.

The asymptotic form of the amplitude at higher orders, and thus its superficial degree of divergence, follows from a simple power-counting argument: with q the loop-momentum, the non-relativistic nucleon-propagator scales asymptotically as $1/q^2$, its non-relativistic kinetic energy as q^2/M, and a loop integral counts as q^5/M. The deuteron propagator (2.1) approaches $1/q$. Only corrections at N^nLO which are proportional to positive powers

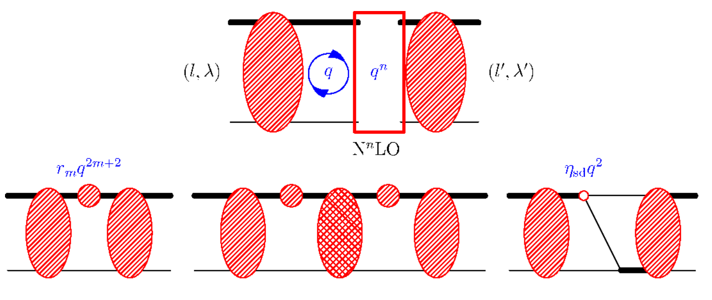

Fig. 2. Top: generic loop correction (rectangle) to the Nd scattering amplitude at N^nLO, proportional to q^n. Bottom, left to right: exemplary higher-order contributions to Nd scattering from the effective-range expansion (blob) and SD-mixing (circle). Hatched ellipse: full off-shell amplitude. Notice that the external legs are on-shell.

of q are relevant in the UV-limit—all other corrections do not modify the leading-order asymptotics. They appear together with some coefficients C which encode short-distance phenomena and whose magnitude is hence set by the breakdown scale of the theory. Such corrections scale asymptotically as $(q/\Lambda_{\not\pi})^n \sim Q^n$ for dimensional reasons. The asymptotics of the generic N^{n}LO-correction to the LO amplitude represented by the rectangle in the top graph of Fig. 2 is thus proportional to

$$k^l q^{-(s_l(\lambda)+1)} \times \frac{q^5}{M}\frac{M}{q^2}\frac{1}{q}\left(\frac{q}{\Lambda_{\not\pi}}\right)^n \times k^{l'} q^{-(s_{l'}(\lambda')+1)} \propto k^l k^{l'} q^{n-s_l(\lambda)-s_{l'}(\lambda')}. \tag{2.11}$$

We therefore identify $n - s_l(\lambda) - s_{l'}(\lambda')$ as the *superficial degree of divergence* of a diagram. A correction at N^{n}LO diverges when

$$\mathrm{Re}\big[n - s_l(\lambda) - s_{l'}(\lambda')\big] \geqslant 0. \tag{2.12}$$

While $s_l(\lambda)$ will turn out to be generically complex, this condition depends only on its real part. The power n of the higher-order insertion is on the other hand a positive integer. Notice also that this formula is not limited to scattering of three nucleons—it applies equally well when external currents couple to nucleons inside the box of Fig. 2.

Usually, higher-order corrections mix different partial waves, and also Wigner-symmetric and Wigner-antisymmetric amplitudes in the doublet-channel. They come from any combination of the following effects, some of which are depicted in the lower panel of Fig. 2:

(1) Effective-range corrections to the deuteron propagator,

$$D(q_0,\vec{q}) \to \frac{1}{\gamma - \sqrt{\frac{\vec{q}^2}{4} - Mq_0}} \times \left[\sum_{m=0}^{\infty} \frac{\frac{r_0}{2}\big(Mq_0 - \frac{\vec{q}^2}{4}\big) + \sum_{n=1}^{\infty} r_n \big(Mq_0 - \frac{\vec{q}^2}{4}\big)^{n+1}}{\gamma - \sqrt{\frac{\vec{q}^2}{4} - Mq_0}}\right]^m. \tag{2.13}$$

With the coefficients $r_n \sim 1/\Lambda_{\not\pi}^{2n+1}$ of natural size, these contributions are ordered by powers of $Q \sim q \sim \sqrt{Mq_0}$: the effective range r_0 enters at NLO as one insertion into the scattering amplitude; r_0^n at N^{n}LO as n insertions, etc. The correction r_n starts contributing with one insertion at N^{2n+1}LO. They modify the UV-limit of D from $1/q$ in (2.6) at N^{n}LO to q^{n-1}.

(2) As two-nucleon forces are non-central, different two-nucleon partial waves mix, e.g., the $^3S_1$- and $^3D_1$-waves. This leads to mixing and spitting of partial waves also in the three-body problem, so that in general $s_l(\lambda) \neq s_{l'}(\lambda')$. By parity conservation, the lowest-order mixing appears for $l' = l \pm 2$.

(3) The local vertex of two nucleons scattering via higher partial waves $L > 0$ contains $2L$ positive powers of q and mixes partial waves as well.

(4) Insertions into D which correct for the explicit breaking of Wigner $SU(4)$ symmetry are proportional to the differences in the effective-range coefficients of the $^1S_0$- and $^3S_1$-system, e.g., to $\gamma_s - \gamma_t$ or $q^2(\rho_{0,s} - \rho_{0,t})$, see also [8]. This leads to mixtures with $l = l'$ but $\lambda \neq \lambda'$.

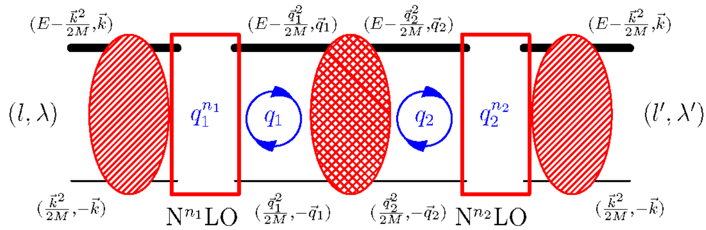

Fig. 3. Exemplary graph containing the full off-shell amplitude. Its kinematics is defined after the integrations over the energy-variables $q_{1,0}$ and $q_{2,0}$ are performed.

(5) Other corrections of less importance, like relativistic corrections to the deuteron propagator D and to the nucleon propagator.

A longer remark is appropriate for corrections in which the LO full off-shell amplitude $t_\lambda^{(l)}$ is sandwiched between two loop-momenta q_1, q_2. They lead to overlapping divergences in these two variables. Examples are given in the centre bottom graph of Fig. 2, or for a correction at N$^{n_1+n_1}$LO involving one full off-shell amplitude in Fig. 3. Its asymptotics is for each off-shell momentum determined by the same Faddeev equation (2.2) with the only difference that the cm-energy E and cm-momentum k become independent variables, with the on-shell point at $k = p = \sqrt{4(ME+\gamma^2)/3}$. In the asymptotic region $E \ll k, p$, the integral equations for both off-shell momenta in the kinematics defined in Fig. 3 are hence in analogy to (2.9) given by:

$$\begin{aligned} t_\lambda^{(l)}(k,p) &= 8\pi\lambda\frac{(-1)^l}{kp}Q_l\left(\frac{p}{k}+\frac{k}{p}\right)+\frac{8\lambda}{\sqrt{3}\pi}(-1)^l\int_0^\infty\frac{\mathrm{d}q}{p}Q_l\left(\frac{p}{q}+\frac{q}{p}\right)t_\lambda^{(l)}(k,q) \\ &= 8\pi\lambda\frac{(-1)^l}{kp}Q_l\left(\frac{p}{k}+\frac{k}{p}\right)+\frac{8\lambda}{\sqrt{3}\pi}(-1)^l\int_0^\infty\frac{\mathrm{d}q}{k}Q_l\left(\frac{k}{q}+\frac{q}{k}\right)t_\lambda^{(l)}(q,p). \end{aligned} \tag{2.14}$$

It is symmetric under the interchange of k and p. In analogy to the solution of the half off-shell amplitude in the next subsection, its asymptotics is constructed in Appendix A.3 as:

$$t_\lambda^{(l)}(k,p) \propto \left\{\begin{array}{ll} \frac{1}{kp}\left(\frac{k}{p}\right)^{s_l(\lambda)} & \text{for } p>k \\ \frac{1}{kp}\left(\frac{p}{k}\right)^{s_l(\lambda)} & \text{for } p<k \end{array}\right\} \propto \frac{1}{kp}. \tag{2.15}$$

The latter follows by analytic continuation to $p \sim k \gg E, \gamma$. One may motivate this result by observing that in the absence of any other scale, this is the result with both the correct mass-dimensions and symmetry properties. It is independent of the angular momentum and spin-parameter. The superficial degree of divergence of Fig. 3 is now easily determined for $q \sim q_1 \sim q_2$ scaling alike:

$$k^l q_1^{-(s_l(\lambda)+1)} \times \frac{q_1^5}{M}\frac{M}{q_1^2}\frac{1}{q_1}\left(\frac{q_1}{\Lambda_{\not\pi}}\right)^{n_1} \times \frac{1}{q_1 q_2} \times \frac{q_2^5}{M}\frac{M}{q_2^2}\frac{1}{q_2}\left(\frac{q_2}{\Lambda_{\not\pi}}\right)^{n_2} \times k^{l'} q_2^{-(s_{l'}(\lambda')+1)}$$
$$\sim k^l k^{l'} q^{n_1+n_2-s_l(\lambda)-s_{l'}(\lambda')}. \tag{2.16}$$

The overlapping divergence $E, k \ll q_1, q_2$ is hence also included in the previous estimate (2.12) when $n = n_1 + n_2$. One readily generalises to any number j of insertions of full off-shell amplitudes with two-nucleon interactions at N^{n_i}LO, $i = 1, \ldots, j+1$, between them and the initial and final half off-shell amplitudes. They all are covered by (2.12), with the higher-order correction at N^nLO, $n = \sum_{i=1}^{j+1} n_i$.

None of the corrections listed above leads at a given order N^nLO to stronger modifications of the short-distance asymptotics than those induced by effective-range corrections entering at the same order, and every order contains also contributions from effective-range corrections.

2.3. Short-distance asymptotics of the amplitude

We now just have to determine the unphysical short-distance behaviour of the amplitude to infer from (2.12) at which order the first three-body forces are needed to absorb divergences. Naïvely, $t_\lambda^{(l)}(p)$ should have the same asymptotics as each of the individual diagrams which need to be summed at LO, see top row of Fig. 1. That means, the asymptotic form should be given by the inhomogeneous or driving term as

$$t_\lambda^{(l)}(p) \propto \lim_{k\to 0} \mathcal{K}^{(l)}\left(\frac{3k^2}{4}; k, p\right) \propto \frac{k^l}{p^{l+2}}, \quad \text{i.e.,} \quad s_{l,\text{simplistic}}(\lambda) = l+1. \tag{2.17}$$

This “simplistic” application of a naïve dimensional estimate reflects the expectation that three-body forces should enter only at high orders, and that the asymptotics in higher partial waves should be suppressed by a centrifugal barrier. Indeed, this estimate would lead from (2.12) to the finding that the three-body force in the lth partial wave—containing at least $2l$ derivatives—occurs only at N^{2l+2}LO and is in particular independent of the spin-parameter λ. However, the three-body problem consists already at leading order of an infinite number of graphs, see Fig. 1. As is well-explored for S-waves, this modifies the solution drastically.

The integral equation (2.9) can be solved exactly by a Mellin transformation since its homogeneous term is scale-invariant and inversion-symmetric; see Appendix A for details. An implicit, transcendental, algebraic equation determines the asymptotic exponent $s_l(\lambda)$:

$$1 = (-1)^l \frac{2^{1-l}\lambda}{\sqrt{3\pi}} \frac{\Gamma\left[\frac{l+s+1}{2}\right]\Gamma\left[\frac{l-s+1}{2}\right]}{\Gamma\left[\frac{2l+3}{2}\right]} {}_2F_1\left[\frac{l+s+1}{2}, \frac{l-s+1}{2}; \frac{2l+3}{2}; \frac{1}{4}\right]. \tag{2.18}$$

It depends only on λ and l. The function ${}_2F_1[a,b;c;x]$ is the hyper-geometric series [27]. This formula comprises the main mathematical result of this article, extends Danilov’s result for $l = 0$ [10], and forms in particular the base to power-count all three-body forces. However, not all of its solutions solve also the integral equation: while both s and $-s$ are together with their complex conjugates solutions to the algebraic equation, only those amplitudes which converge for $p \to \infty$ and for which the Mellin transformation exists are

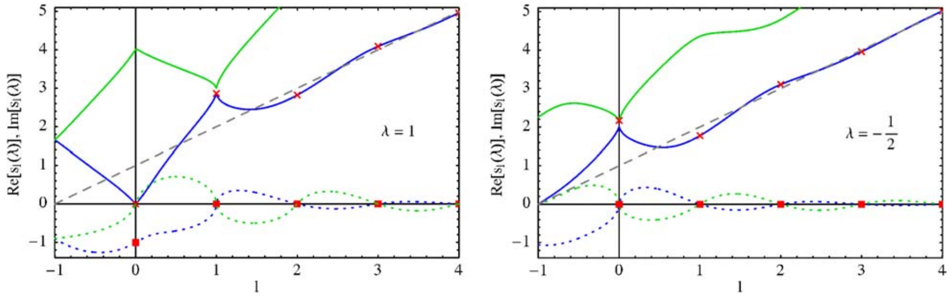

Fig. 4. The first two solutions $s_l(\lambda)$ at $\lambda = 1$ (left) and $\lambda = -1/2$. Solid (dotted): real (imaginary) part; dashed: simplistic estimate (2.17); cross (square): real (imaginary) part of the asymptotics obtained by a fit of the full solution to the Faddeev equation (2.2) at large off-shell momenta. Dark/light: first/second solution. An Efimov effect occurs only for $|\,\mathrm{Re}[s]| < \mathrm{Re}[l+1]$, i.e., when the solid line lies below the dashed one, and $\mathrm{Im}[s] \neq 0$.

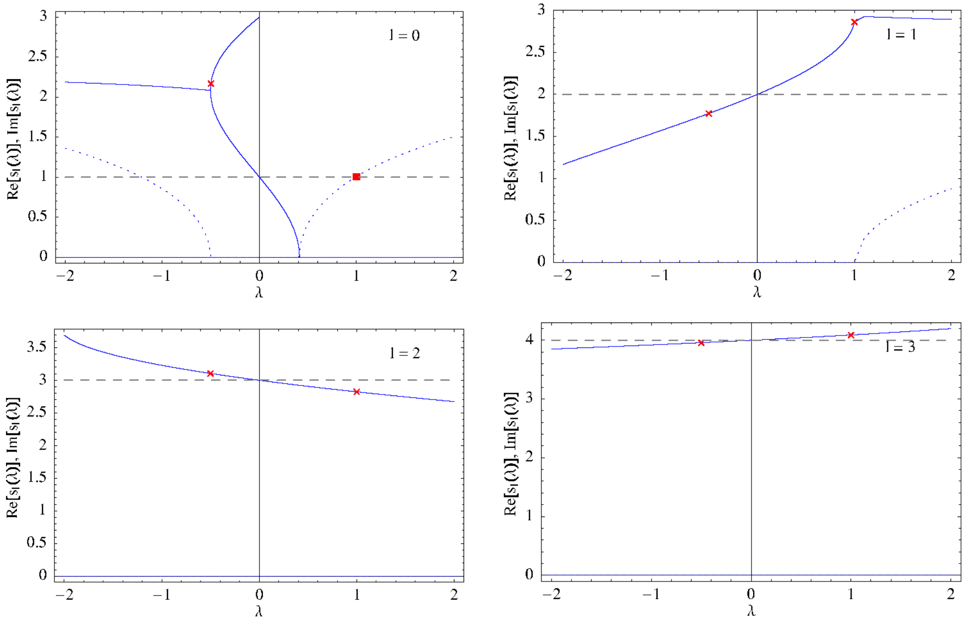

Fig. 5. $s_l(\lambda)$ at $l = \{0; 1; 2; 3\}$. Notation as in Fig. 4.

permitted. Most notably, this constrains $\mathrm{Re}[s] > -1$, $\mathrm{Re}[s] \neq \mathrm{Re}[l] \pm 2$; see Appendix A. Furthermore, out of the infinitely many, in general complex solutions for given l and λ, only the one survives as relevant in the UV-limit whose real part is closest to -1, i.e., for which $\mathrm{Re}[s_l(\lambda) + 1]$ is minimal. We consider in the following only those solutions which match these criteria.

Plots of one of the values in the quadruplet of two-parameter functions $\{\pm s_l(\lambda), \pm s_l^*(\lambda)\}$ at fixed l and fixed λ, respectively, are given in Figs. 4 and 5. Table 2 lists the first $s_l(\lambda)$ for the partial waves $l \leqslant 4$ and $\lambda = \{-1/2; 1\}$, compared to the simplistic estimate (2.17).

Table 2
Solutions $s_l(\lambda)$ to (2.18) for the most relevant physical systems

Partial wave l	$s_l(\lambda=1)$	$s_l(\lambda=-1/2)$	$s_{l,\text{simplistic}} = l+1$
0	1.00624... i	2.16622...	1
1	2.86380...	1.77272...	2
2	2.82334...	3.10498...	3
3	4.09040...	3.95931...	4
4	4.96386...	5.01900...	5

Let us for the remainder of this subsection investigate the rich structure of this result. Branch points occur, e.g., for $(l=0; \lambda \approx -1/2)$, where imaginary parts open. Avoided crossings are found, e.g., for $(l=1; \lambda=1)$, etc. While the solution is in general complex, it is real for non-negative integer l in the physical channels discussed above, where $\lambda = \{1; -1/2\}$. The only exception is the imaginary solution for $(l=0; \lambda=1)$ first found by Danilov [10]. It makes a three-body force in this channel mandatory already at LO as the system would otherwise be unstable against collapse of its wave-function to the origin, a phenomenon well-known to be related to the Thomas and Efimov effects [14,30] and giving rise to a limit-cycle [11–13] manifesting itself in the Phillips line [6,31,32]. Its interpretation is not the scope of this presentation; we only note that the power-counting of three-body forces in this channel states that a new, independent three-body force with $2l$ derivatives enters at N^{2l}LO [8]. It must be seen as coincidence that the naïve dimensional estimate in (2.12)—where $n>0$ was assumed explicitly—leads to the same conclusion.

In general, s can be complex, as, for example, at $(l=0; \lambda<-1/2)$, $(l=1; \lambda>1)$ or l non-integer, $\lambda = \{1; -1/2\}$. In that case, out of the four independent solutions, the ones with $\text{Re}[s] \leqslant -1$ must be eliminated as $t(p)$ does not converge for them. In cases like $(l \in [-0.5819\ldots; 0.3446\ldots]; \lambda=1)$ where all four complex solutions obey $\text{Re}[s] > -1$, only the solutions with minimal $\text{Re}[s_l(\lambda)+1]$ survive, as shown above. These remaining two solutions $s := s_R \pm \mathrm{i} s_I$ are equally strong and must be super-imposed:

$$t_\lambda^{(l)}(p) \propto \frac{\sin[s_I \ln[p] + \delta]}{p^{s_R+1}}. \tag{2.19}$$

Usually, Fredholm's alternative forbids that both the homogeneous and inhomogeneous integral equations have simultaneous solutions. Therefore, the boundary conditions of the integral equation fix the phase δ to a unique value. However, when the kernel of the Faddeev equation (2.2) is singular, this operator has no inverse and Fredholm's theorem does not apply. For this to occur, the solution to the integral equation must be unique only up to a zero-mode of the homogeneous version, i.e.,

$$\frac{8\lambda}{\sqrt{3}\pi}(-1)^l \int_0^\infty \frac{\mathrm{d}q}{p} Q_l\left(\frac{p}{q} + \frac{q}{p}\right) a_\lambda^{(l)}(q) = a_\lambda^{(l)}(p), \tag{2.20}$$

must have a non-trivial solution $a_\lambda^{(l)}(q) \not\equiv 0$. As the explicit construction in Appendix A.2 demonstrates, this is the case if and only if $\text{Re}[l+1] > |\text{Re}[s]|$, i.e., when $\text{Re}[s]$ is smaller in magnitude than the blind estimate $s_{l,\text{simplistic}}$, (2.17). In that case, a one-parameter family of solutions arises. A three-body force is then necessary not to cure divergences but to

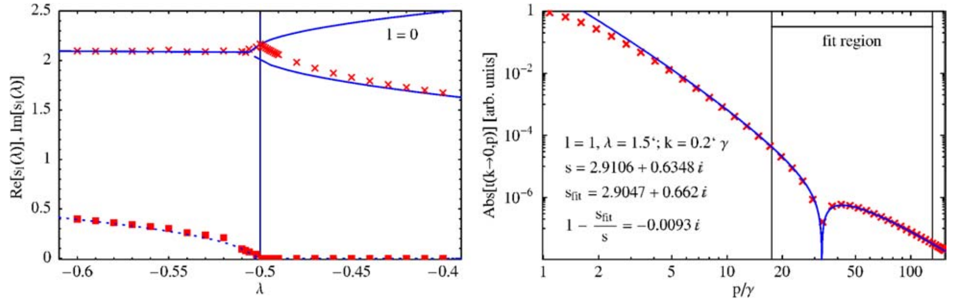

Fig. 6. Left: numerical and analytical solution for $s_l(\lambda)$ at $l = 0$ around $\lambda = -1/2$. Right: numerical determination of $s_l(\lambda)$, exemplified for $l = 1$, $\lambda = 1.5$, comparing "data" (crosses) and the fitted function (2.19) (solid line). Notation as in Fig. 4.

absorb the dependence on the free parameter δ. Its initial condition is not constrained by two-body physics but must be determined by a three-body datum. Thus, one finds a limit-cycle for such systems, like for three spinless bosons or the Wigner-symmetric part of the $^2S_{1/2}$-wave amplitude in Nd scattering, $(l = 0; \lambda = 1)$, discussed above. As easily read-up from Figs. 4 and 5, this phenomenon occurs for non-negative integer angular momentum only when $l = 0$ and $\lambda > 3\sqrt{3}/(4\pi)$, where $\mathrm{Re}[s] = 0$. However, an Efimov effect with complex s is often found for non-integer l, e.g., for $l \in [-0.3544\ldots; 0.5452\ldots]$ in the three-boson case, $\lambda = 1$. A closer investigation will be interesting in view of a conjecture on regularising the three-body system in Section 3.2.

A numerical investigation of the Faddeev equation (2.2) confirms these findings. In order to compute a solution, one introduces a cut-off Λ which is unphysical and thus not to be confused with the breakdown scale $\Lambda_{\not\pi}$ of the EFT. The numerical values of $s_l(\lambda)$ are found from fitting the half off-shell amplitude $t_\lambda^{(l)}(E, k; p)$ at $E, k, \gamma \ll p \ll \Lambda$ to the asymptotic forms (2.10) and (2.19). A grid of 100 points is easily enough for a numerical precision in s of about 1%.[1] Agreement between the numerical and analytical solution is excellent also at non-integer l and $\lambda \neq \{1, -1/2\}$, see Fig. 6 besides Figs. 4 and 5 for examples. Particularly interesting is in that context the neighbourhood around the $(\lambda = -\frac{1}{2})$-solution in the S-wave channel, $l = 0$. Here, the first solution to the algebraic equation (2.18) is $s = 2$, but the Mellin transform does not exist at that point because $\mathrm{Re}[s] = l \pm 2$. The system is here also close to the branch-point at $(\lambda = -0.50416\ldots; s \approx 2.0836\ldots)$, where an imaginary part opens for smaller λ. Another branch-point lies at $(l = 1; \lambda = 1.0053\ldots; s = 2.93164\ldots)$.

To summarise, the algebraic equation (2.18) for $s_l(\lambda)$ provides asymptotic solutions of the form (2.10) to the three-body Faddeev equation (2.2) for $\mathrm{Re}[s] > -1$ and $\mathrm{Re}[s] \neq \mathrm{Re}[l \pm 2]$. Only those solutions are relevant in the UV-limit $p \gg \gamma, E, k$ for which $\mathrm{Re}[s+1]$ is minimal. The Efimov effect occurs only if $\mathrm{Im}[s] \neq 0$ and $|\mathrm{Re}[s]| < \mathrm{Re}[l + 1]$, because only then is the kernel of the integral equation not compact.

[1] A simple MATHEMATICA code can be downloaded from http://www.physik.tu-muenchen.de/~hgrie.

2.4. Ordering three-body forces

Although divergences can occur as soon as the superficial degree of divergence $\mathrm{Re}[n - s_l(\lambda) - s_{l'}(\lambda')] \geqslant 0$, only those are physically meaningful which can be absorbed by three-body counter-terms, i.e., by a local interaction between three nucleons in the given channels. Naïve dimensional analysis does not construct the three-body forces. It thus predicts some divergences which are absent when the diagram is actually calculated. There is, for example, no Wigner-$SU(4)$-antisymmetric three-body force without derivatives [6], so that the divergence must in this case be at least quadratic for infinite cut-off, $\mathrm{Re}[n - s_l(\lambda) - s_{l'}(\lambda')] \geqslant 2$. For finite cut-off Λ, a three-body force without derivatives can be constructed which is non-local on a scale smaller than $1/\Lambda$—but appears of course local at scales smaller than the break-down scale $\Lambda_{\not\pi} \lesssim \Lambda$ of EFT($\not\pi$). As its coefficient disappears when the cut-off is sent to infinity, it is in a renormalisation group analysis classified as "irrelevant". Except for this example which is relevant below, this article will, however, simply assume that the first three-body force enters with the first divergence. To construct these forces in detail is left for future, more thorough investigations.

The solution of (2.18) approaches for large integer l in the physically most interesting cases $\lambda = \{1; -1/2\}$ the simplistic estimate of (2.17): $s_{\text{simplistic}} = l + 1$; see Table 1 and Figs. 4, 5. This reflects that the Faddeev equation should be saturated by the Born approximation in the higher partial waves because of the ever-stronger centrifugal barrier between the deuteron and the nucleon. Therefore, three-body forces enter in most channels for all practical purposes at the same order as suggested by the simplistic argument, namely $\mathrm{N}^{l+l'+2}$LO between the lth and l'th partial waves. It is therefore convenient to introduce the variable

$$\Delta_l(\lambda) := s_l(\lambda) - (l + 1), \tag{2.21}$$

which parameterises how strongly simplistic and actual asymptotic form differ. For $\Delta > 0$, the superficial degree of divergence of the LO amplitude is weaker than guessed by (2.17).

In the lower partial waves $l, l' \leqslant 2$, however, the blind expectation deviates substantially from the exact solution; see Table 3. For $(l = 0; \lambda = 1)$, for example, $s_{l,\text{simplistic}} = l + 1$ under-estimates the short-distance asymptotics of $t(p)$, while $s_0(1) = 1.006\ldots\mathrm{i}$ is even imaginary. A limit-cycle signals that one must include a three-body force already at LO, as briefly hinted upon above. Multiple insertions of three-body forces are not suppressed.

For two partial waves, the formula (2.17) substantially *over-estimates* the asymptotics of $t_\lambda^{(l)}$. Therefore, a three-body force is in channels which involve these partial waves *weaker* than predicted by a simplistic application of naïve dimensional analysis. Consider first the case of three bosons with $(l = 1; \lambda = 1)$: $s = 2.86\ldots > s_{\text{simplistic}} = 2$, $\Delta = 0.86\ldots$. While the first divergence from the two-body sector arises in this partial wave at $\mathrm{N}^{5.72}$LO, one would—following (2.17)—have predicted the first three-body force as necessary already at N^4LO. It is in this channel hence demoted by ≈ 1.7 orders.

The situation is even more drastic in the ${}^4\mathrm{S}_{3/2}$-channel of Nd scattering, $(l = 0; \lambda = -1/2)$: here, only divergences which are at least quadratic are physical because the first three-body force must contain at least two derivatives since the Pauli principle forbids a momentum-independent three-nucleon force—or equivalently, no Wigner-$SU(4)$-

Table 3
Order n_0 at which the leading three-nucleon force enters for the lowest channels $l, l' \leqslant 2$, comparing the simplistic estimate (2.17) and the actual values (2.12/2.18). The list follows the physical partial wave mixing, and the subscript Ws (Wa) denotes the Wigner-symmetric (antisymmetric) contribution. In the ${}^2\mathrm{S}_{\mathrm{Wa}}$- and ${}^4\mathrm{S}_{3/2}$-channels, the absence of a three-nucleon force without derivatives is taken into account by the factor "+2". The last column indicates whether the three-body force is stronger ("promoted") or weaker ("demoted") than the simplistic estimate suggests. When the difference between the two is in magnitude smaller than 0.5, they are quite arbitrarily assumed to enter at the same order

Channel			Estimate	Simplistic	
$(\lambda; l)$	$(\lambda'; l')$	Partial waves	$\mathrm{Re}[s_l(\lambda) + s_{l'}(\lambda')]$	$l + l' + 2$	
$(1; 0)$	$(1; 0)$	${}^2\mathrm{S}_{\mathrm{Ws}}$–${}^2\mathrm{S}_{\mathrm{Ws}}$	LO	$\mathrm{N}^2\mathrm{LO}$	promoted
$(1; 0)$	$(-\frac{1}{2}; 0)$	${}^2\mathrm{S}_{\mathrm{Ws}}$–${}^2\mathrm{S}_{\mathrm{Wa}}$	$\mathrm{N}^{2.2+2}\mathrm{LO}$	$\mathrm{N}^{2+2}\mathrm{LO}$	
$(-\frac{1}{2}; 0)$	$(-\frac{1}{2}; 0)$	${}^2\mathrm{S}_{\mathrm{Wa}}$–${}^2\mathrm{S}_{\mathrm{Wa}}$	$\mathrm{N}^{4.3+2}\mathrm{LO}$		demoted
$(1; 0)$	$(-\frac{1}{2}; 2)$	${}^2\mathrm{S}_{\mathrm{Ws}}$–${}^4\mathrm{D}$	$\mathrm{N}^{3.1}\mathrm{LO}$	$\mathrm{N}^4\mathrm{LO}$	promoted
$(-\frac{1}{2}; 0)$	$(-\frac{1}{2}; 2)$	${}^2\mathrm{S}_{\mathrm{Wa}}$–${}^4\mathrm{D}$	$\mathrm{N}^{5.3}\mathrm{LO}$		demoted
$(1; 1)$	$(1; 1)$	${}^2\mathrm{P}_{\mathrm{Ws}}$–${}^2\mathrm{P}_{\mathrm{Ws}}$	$\mathrm{N}^{5.7}\mathrm{LO}$		demoted
$(1; 1)$	$(-\frac{1}{2}; 1)$	${}^2\mathrm{P}_{\mathrm{Ws}}$–${}^2\mathrm{P}_{\mathrm{Wa}}$, ${}^2\mathrm{P}_{\mathrm{Ws}}$–${}^4\mathrm{P}$	$\mathrm{N}^{4.6}\mathrm{LO}$	$\mathrm{N}^4\mathrm{LO}$	demoted
$(-\frac{1}{2}; 1)$	$(-\frac{1}{2}; 1)$	${}^2\mathrm{P}_{\mathrm{Wa}}$–${}^2\mathrm{P}_{\mathrm{Wa}}$, ${}^4\mathrm{P}$–${}^4\mathrm{P}$	$\mathrm{N}^{3.5}\mathrm{LO}$		
$(-\frac{1}{2}; 0)$	$(-\frac{1}{2}; 0)$	${}^4\mathrm{S}$–${}^4\mathrm{S}$	$\mathrm{N}^{4.3+2}\mathrm{LO}$	$\mathrm{N}^{2+2}\mathrm{LO}$	demoted
$(-\frac{1}{2}; 0)$	$(1; 2)$	${}^4\mathrm{S}$–${}^2\mathrm{D}_{\mathrm{Ws}}$	$\mathrm{N}^{5.0}\mathrm{LO}$	$\mathrm{N}^4\mathrm{LO}$	demoted
$(-\frac{1}{2}; 0)$	$(-\frac{1}{2}; 2)$	${}^4\mathrm{S}$–${}^2\mathrm{D}_{\mathrm{Wa}}$, ${}^4\mathrm{S}$–${}^4\mathrm{D}$	$\mathrm{N}^{5.3}\mathrm{LO}$		demoted
$(1; 2)$	$(1; 2)$	${}^2\mathrm{D}_{\mathrm{Ws}}$–${}^2\mathrm{D}_{\mathrm{Ws}}$	$\mathrm{N}^{5.6}\mathrm{LO}$		
$(1; 2)$	$(-\frac{1}{2}; 2)$	${}^2\mathrm{D}_{\mathrm{Ws}}$–${}^2\mathrm{D}_{\mathrm{Wa}}$, ${}^2\mathrm{D}_{\mathrm{Ws}}$–${}^4\mathrm{D}$	$\mathrm{N}^{5.9}\mathrm{LO}$	$\mathrm{N}^6\mathrm{LO}$	
$(-\frac{1}{2}; 2)$	$(-\frac{1}{2}; 2)$	${}^2\mathrm{D}_{\mathrm{Wa}}$–${}^2\mathrm{D}_{\mathrm{Wa}}$, ${}^4\mathrm{D}$–${}^4\mathrm{D}$	$\mathrm{N}^{6.2}\mathrm{LO}$		

antisymmetric three-body force exists [6]. Therefore, the divergence condition (2.12) reads $\mathrm{Re}[n - s_l(\lambda) - s_{l'}(\lambda')] \geqslant 2$. Since $s = 2.16\ldots > s_{\mathrm{simplistic}} = 1$, the first three-body force enters thus not at $\mathrm{N}^4\mathrm{LO}$ but at least two orders higher, namely at $\mathrm{N}^{6.33\cdots}\mathrm{LO}$.

As an example for mixing between partial waves, consider the ${}^4\mathrm{S}_{3/2}$-wave: it mixes with both the ${}^4\mathrm{D}_{3/2}$-wave ($\lambda = -1/2$) and the Wigner-symmetric and antisymmetric components of the ${}^2\mathrm{D}_{3/2}$-wave ($\lambda = 1$ or $-1/2$). All of them are already close to the estimate $s_{l,\mathrm{simplistic}} = l + 1 = 3 > s_{l=0}(\lambda = -1/2)$. Still, the first divergences induced by this mixing start from (2.12) at $\mathrm{N}^{\approx 5}\mathrm{LO}$, i.e., approximately one order higher than blindly guessed.

More modifications induced by mixing and splitting of partial waves as well as explicit breaking of the Wigner $SU(4)$ symmetry are straight-forwardly explored, but left to a future publication. Table 3 summarises the findings for the physically most relevant three-body channels $l, l' \leqslant 2$. Except in the ${}^4\mathrm{S}_{3/2}$-channel, it does not take into account whether a three-body counter-term can actually be constructed at the order at which the first divergence occurs. However, while this can make a three-body force occur at a higher *absolute* order than listed, the *relative* demotion of a three-body force to higher orders by modifications of the superficial degree of divergence holds.

"Fractional orders" are a generic feature of (2.18), combined with (2.12). Consider again as example the case $(l = 0; \lambda = -1/2)$, where the first two-body divergence appears formally at $n = 6.33\ldots$, while including a two-body correction with fractional order is of course impossible: clearly, the N^7LO amplitude diverges without three-body forces, but one could also argue that it is prudent to include a three-body force already at N^6LO because the higher-order correction to the amplitude converges only weakly, namely as $q^{-0.33}$. Therefore, the integral over q becomes unusually sensitive to the amplitude at large off-shell momenta, above the breakdown scale $\Lambda_{\not\pi}$ of the EFT, and therefore to details of physics at distances on which the EFT is not any more valid. In the more drastic case $s_{l=4}(\lambda = -1/2) = 5.02\ldots$, $\Delta = 0.02\ldots$, the first three-body force enters following (2.12) at $\mathrm{N}^{10.04}$LO. Therefore, no divergence arises in the two-body sector before N^{11}LO, but it seems reasonable to include it already at N^{10}LO because the higher-order corrections from two-nucleon insertions converge then generically to a cut-off independent result only very slowly, namely as $q^{-0.04}$.

Naïve dimensional analysis cannot decide the question at which order precisely a "fractional divergence" gives rise to a three-body force as it argues on a diagram-by-diagram basis, missing possible cancellations between different contributions at the same order. One way to settle it is to see whether the cut-off dependence of observables follows the pattern required in EFT. Recall that N^nLO corrections contribute to observables typically as

$$Q^n = \left(\frac{p_{\mathrm{typ}}}{\Lambda_{\not\pi}}\right)^n \tag{2.22}$$

compared to the LO result and that low-energy observables must be independent of an arbitrary regulator Λ up to the order of the expansion. In other words, the physical scattering amplitude must be dominated by integrations over off-shell momenta q in the region in which the EFT is applicable, $q \lesssim \Lambda_{\not\pi}$. As argued, e.g., by Lepage [33], one can therefore estimate sensitivity to short-distance physics, and hence provide a reasonable error-analysis, by employing a momentum cut-off Λ in the solution of the Faddeev equation and varying it between the breakdown scale $\Lambda_{\not\pi}$ and ∞. If observables change over this range by "considerably" more than Q^{n+1}, a counter-term of order Q^n should be added. This method is frequently used to check the power-counting and systematic errors in EFT($\not\pi$) with three nucleons, see, e.g., most recently [19]. A similar argument was also developed in the context of the EFT "with pions" of nuclear physics [34,35]. Such reasoning goes, however, beyond the clear prescription according to which only divergences make the inclusion of counter-terms mandatory and opens the way to a softer criterion—which is obviously formulated rigorously only with great difficulty. How to treat "fractional orders" in a well-prescribed and consistent way must thus be investigated further.

2.5. How three-body forces run

Before turning to practical consequences of these observations, let us for a moment investigate how the strengths of three-body forces have to scale with q in order to absorb

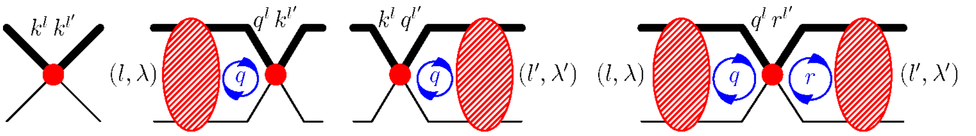

Fig. 7. Generic leading corrections from three-body forces which can be rewritten as Nd-interactions with at least $l + l'$ derivatives.

the divergences (2.11) from two-body interactions. In contradistinction to the above considerations where the specific form of the three-body force did not enter, we now limit the discussion to those three-body forces which can be rewritten as deuteron–nucleon interactions. Clearly, all three-body forces which are needed to absorb divergences from two-nucleon effective-range corrections proportional to r_n fall into that class.[2] As discussed in Section 2.2, this is no severe restriction because every divergence contains such a piece. The leading contributions are given by the diagrams of Fig. 7. At even higher orders, two- and three-body corrections occur simultaneously in one graph. Let the three-body force between deuteron and nucleon with relative incoming momentum p and outgoing momentum p' scale as

$$p^l p^{l'} \frac{h}{\Lambda_{\not\pi}^{l+l'+2}}, \tag{2.23}$$

where the dimensionless coupling h encodes the short-distance details of three interacting nucleons which are not resolved in EFT($\not\pi$). It absorbs hence also the divergences generated at order $s_l(\lambda) + s_{l'}$ from two-body interactions, (2.11), to render the result at this order insensitive of unphysical short-distance effects. The parameter h must thus formally scale as q to some power α which is determined such that at least one of the three-body force graphs appears at the same order as the divergence. The graphs scale and diverge as:

$$\begin{array}{lll}
\text{(a)} & \sim k^l k^{l'} q^{\alpha}, & \text{no divergence,} \\
\text{(b1)} & \sim k^l k^{l'} q^{\alpha-\Delta_l(\lambda)}, & \text{diverges for } \mathrm{Re}\big[\Delta_l(\lambda)\big] \leqslant 0, \\
\text{(b2)} & \sim k^l k^{l'} q^{\alpha-\Delta_{l'}(\lambda')}, & \text{diverges for } \mathrm{Re}\big[\Delta_{l'}(\lambda')\big] \leqslant 0, \\
\text{(c)} & \sim k^l k^{l'} q^{\alpha-\Delta_l(\lambda)-\Delta_{l'}(\lambda')}, & \text{diverges for } \mathrm{Re}\big[\Delta_l(\lambda)\big] \leqslant 0 \text{ or } \mathrm{Re}\big[\Delta_{l'}(\lambda')\big] \leqslant 0.
\end{array} \tag{2.24}$$

The tree-level contribution is of course free of divergences. Notice that the three-body force h absorbs for non-integer s also the non-analytic piece of the divergence (2.11), and, in particular, the phase δ when $\mathrm{Im}[s] \neq 0$, see (2.19). This piece is non-analytic in the unphysical off-shell momentum q, but of course analytic in the low-energy momentum k.

The graphs containing three-body forces enter at the same order for all channels which follow the simplistic estimate of (2.17), i.e., $\Delta_l(\lambda), \Delta_{l'}(\lambda') = 0$. Then, all three-body corrections (a)–(c) occur at the same order α, the three-body force counts as $h \sim q^{\alpha} = q^{l+l'+2}$,

2 But not necessarily three-body forces which contribute to the mixing and splitting of partial waves.

and the logarithmic divergences of the loop-diagrams (b1), (b2), (c) are absorbed into h as well. To determine the order at which a three-body force enters, it is therefore sufficient to count in this case its mass-dimension, which is also given by $l + l' + 2$, see (2.23). This is nearly realised in the higher partial waves, where $|\Delta|$ is usually not bigger than 0.3.

For $(l = 0; \lambda = 1)$, however, $\mathrm{Re}[\Delta] = -1$ and $\alpha = 0$. Now, multiple insertions of three-body forces are not suppressed and the Efimov effect mandates including the three-body force in the LO Faddeev equation. The strength h depends on the arbitrary phase δ, showing a limit-cycle [11–13] as manifested in the Phillips line [6,31,32]. Interestingly, the diagrams (a), (b1), (b2)—as well as their analogues with higher-order three-nucleon forces—now become following (2.24) formally corrections of higher order, as found numerically in Ref. [7].

As seen in the previous section, three-body forces appear in many channels at higher orders than expected. This now also regroups the graphs containing three-body forces. According to the scaling properties (2.24), the tree-level diagram (a) is in the $(l = 1; \lambda = 1)$ channel ≈ 1.7 orders weaker than the leading three-body diagram (c) because $\Delta = 0.86\ldots$. It can therefore safely be neglected when absorbing the leading divergences from two-body insertions. The graphs (b1), (b2) are down by $\approx 0.9\ldots$ orders. For $(l = 0; \lambda = -1/2)$, this is even more pronounced: with $\Delta = 1.16\ldots$, the tree-level three-body contribution (a) is suppressed by more than two, and (b1), (b2) by more than one order against the sandwiched three-body graph (c), so that both can be neglected when the leading divergences are absorbed into (c) only. Notice that all three-body corrections converge for $\Delta > 0$.

Possible overlapping divergences complicate a similar analysis in the case of three-body forces which cannot be rewritten as Nd-interactions, warranting further investigations which are however not central to this presentation.

3. Consequences

The first goal of this publication has been reached: Eq. (2.12) is an explicit formula for the order at which the first three-body force must be added to absorb divergences. It depends on the partial wave l and channel λ via the exponent $s_l(\lambda)$ which characterises the asymptotic form of the half off-shell scattering matrix $t_\lambda^{(l)}(E, k; p)$ and is determined by (2.18). A simplistic application of naïve dimensional analysis, (2.17), provides a good estimate for all partial waves $l \geqslant 2$. However, it over-rates three-body forces of the three-nucleon system, for example, in the ${}^4\mathrm{S}_{3/2}$- and ${}^2\mathrm{P}$-channels, while it under-estimates them e.g., in the ${}^2\mathrm{S}_{1/2}$-wave; see the summary in Table 3. In the case of three spinless bosons, the P-wave three-body interaction is weaker, while the S-wave interaction is stronger than the simplistic argument suggests. With these findings, the EFT of three spinless bosons and the pionless version of EFT in the three-nucleon system, EFT($\not\pi$), are self-consistent field theories which contain the least number of counter-terms at each order to ensure renormalisability. Each three-body counter-term gives rise to one subtraction constant which must be determined by a three-body datum. Let us explore in this section some physically relevant results which can be derived from these findings.

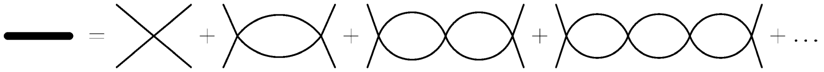

Fig. 8. Resummation of the infinite number of LO two-body diagrams into the deuteron propagator given by (2.1).

3.1. Context

3.1.1. Amending naïve dimensional analysis

As outlined in the introduction, power-counting by naïve dimensional analysis amounts for perturbative theories to little more than counting the mass-dimensions of the interactions [1]. In this case, only a finite number of diagrams contributes at each order. When the LO amplitude is, however, non-perturbative, i.e., an infinite number of diagrams must be summed to produce shallow bound-states, then the situation changes: the LO amplitude can follow for large off-shell momenta a different power-law than the one which one obtains when one considers the asymptotic form of each of the diagrams separately. Then, the "canonical" application turns out to be too simplistic and must be modified as in Section 2.3. This is neither a failure of naïve dimensional analysis, nor should it come completely unexpectedly. An example is indeed already found in the two-body sector of EFT($\not{\pi}$). Recall that an infinite number of two-body scattering diagrams are resummed into the LO deuteron propagator (2.1) to produce the shallow two-body bound state, Fig. 8. Each of the diagrams diverges as q^n, with n the number of loops. Their sum *converges*, however, as $1/q$ for large momenta, see (2.6). In this case, the solution is obtained by a geometric series and the necessary changes are easily taken into account, see the reviews [3–5,17]. What may come as a surprise is that this can also happen when both all LO diagrams separately and their resummation are ultra-violet *finite*. In the three-body case, all diagrams actually show the *same* power-law behaviour $1/q^{l+1}$, see (2.17). However, the resummed form looks very different, exhibiting a non-integer and even complex power-dependence (2.18). This does not occur for every system with shallow bound states. For example, the exchange of a Coulomb photon between two non-relativistic, charged particles in non-relativistic QED scales asymptotically as $1/q^2$. The exact solution of the Coulomb problem has the same scaling behaviour.

3.1.2. External currents

The power-counting of three-body forces developed above applies equally when external currents couple to the three-nucleon system. The only change is that the higher-order interaction in Fig. 2 becomes more involved, introducing also the momentum- or energy-transfer from the external source as additional low-energy scales.

3.1.3. Three-body forces at higher orders

Another trivial extension is to power-count three-body forces beyond the leading ones. In that case, the superficial degree of divergence must by analyticity be larger than a positive even integer, $\mathrm{Re}[n - s_l(\lambda) - s_{l'}(\lambda')] \geqslant 2m$, $m \in 2\mathbb{N}_0$. The higher-order three-body force contains $2m$ derivatives more than the leading one ($m = 0$) and enters $2m$ orders higher. We used this already to power-count the first three-body force in the $^4\mathrm{S}_{3/2}$-channel. The power-counting based on naïve dimensional analysis agrees for the $^2\mathrm{S}_{1/2}$-channel with the one which was recently established by a more careful and explicit construction [6–8].

3.2. Conjectures

3.2.1. Predicting the $^4\mathrm{S}_{3/2}$-scattering length

The first conjecture follows from the observation above that three-body forces are demoted in the $^4\mathrm{S}_{3/2}$-wave from a N^{4}LO-effect by two orders to N$^{6.3}$LO. This has immediate consequences for the quartet S-wave scattering length of the nucleon–deuteron system which has drawn substantial interest recently. Its knowledge sets at present the experimental uncertainty in an indirect determination of the doublet scattering length [36], which in turn is well-known to be sensitive to three-body forces [31]. It was determined repeatedly in EFT($\not\pi$) at N^{2}LO, with different methods to compute higher-order corrections agreeing within the predicted accuracy [21–23,26], e.g., most recently [19]:

$$a\left(^4\mathrm{S}_{3/2}\right) = (\underbrace{5.091}_{\text{LO}} + \underbrace{1.319}_{\text{NLO}} - \underbrace{0.056}_{\text{N}^2\text{LO}})\ \text{fm} = [6.35 \pm 0.02]\ \text{fm}. \tag{3.1}$$

The theoretical accuracy by neglecting higher-order terms is here estimated conservatively by

$$Q \sim \frac{\gamma \approx 45\ \text{MeV}}{\Lambda_{\not\pi} \approx m_\pi} \approx \frac{1}{3}$$

of the difference between the NLO- and N^{2}LO-result. This agrees very well with experiment [37], [6.35 ± 0.02] fm, albeit partial wave mixing, isospin breaking and electromagnetic effects are not present in EFT($\not\pi$) at N^{2}LO. As the amplitude decays at large off-shell momenta as $1/p^{3.16\cdots}$, see Table 1, it is not surprising that $a(^4\mathrm{S}_{3/2})$ is to a very high degree sensitive only to the correct asymptotic tail of the deuteron wave-function. The first three-body force[3] enters not earlier than N^{6}LO—taking a conservative approach as described above to round the "fractional order". Indeed, if the theoretical uncertainty decreases steadily from order to order as it does from NLO to N^{2}LO, then one should be able to reach an accuracy of $\pm(\frac{1}{3})^4 \times 0.06\ \text{fm} \lesssim \pm 0.001\ \text{fm}$ with only two-nucleon scattering data as input—provided those in turn are known with sufficient accuracy. Indeed, this is not much smaller than the range over which modern high-precision potential-model calculations differ: 6.344–6.347 fm [38,39]. To use this number hence as input into a determination of the doublet scattering length as $a(^2\mathrm{S}_{1/2}) = [0.645 \pm 0.003(\text{exp}) \pm 0.007(\text{theor})]\ \text{fm}$ [36] seems justified, and the error induced by the theoretical uncertainty might actually be overestimated. Notice that if the three-nucleon force would occur in EFT($\not\pi$) at N^{4}LO as the simplistic expectation (2.17) suggests, the error should be of the order of $(\frac{1}{3})^2 \times 0.06\ \text{fm} \approx 0.007\ \text{fm}$, considerably larger than the spread in the potential-model predictions.

[3] In the mixing between the $^2\mathrm{S}_{1/2}$-, ^{2}D- and ^{4}D-waves, three-body forces appear already at N^{5}LO, see Table 3, but they are irrelevant for the scattering length.

3.2.2. An alternative regularisation

More speculative is the possibility for a new regularisation scheme. In principle, (2.18) gives the asymptotics $s_l(\lambda)$ of the half off-shell amplitude for arbitrary—even complex—l and λ. As this function is largely analytic, one could use analytic continuation for a "partial wave regularisation" of the three-body system. This is particularly attractive to regulate the limit-cycle problem of the $^2\mathrm{S}_{1/2}$-wave ($l = 0;\ \lambda = 1$), whose practical implications are at present mostly discussed by cut-off regularisation. However, the algebraic solution to (2.18) suffers—as discussed in Section 2.3—from constraints by branch-cuts and regions where the Mellin transformation does not exist. In addition, a limit-cycle is encountered only when s has an imaginary part, and its real part is smaller than the simplistic estimate (2.17). However, the imaginary part disappears in the vicinity of the physical point ($l = 0; \lambda = 1$) only where s has a branch-cut, see Figs. 4 and 5. Blankleider and Gegelia [40] attempted to use analyticity in λ at fixed $l = 0$ to regulate the three-boson problem at LO without resorting to three-body forces to stabilise the system.

3.3. Caveats

Weak points in the derivation should also be summarised:

(1) As always in naïve dimensional analysis, one obtains only the superficial degree of divergence. Except in the $^4\mathrm{S}_{3/2}$-channel, the classification of Table 3 does not take into account whether a three-body counter-term can actually be constructed at the order at which the first divergence occurs. However, while the actual degree of divergence must usually be determined by an explicit calculation, it is never larger than the superficial one. Therefore, a three-body force can possibly occur at a higher *absolute* order than predicted by naïve dimensional analysis, but this applies then equally well to the simplistic estimate. Therefore, the *relative* demotion of a three-body force to higher orders by the modified superficial degree of divergence holds.
In this context, the three-body forces which can contribute in a given channel, and in particular to partial-wave mixing, should be constructed explicitly. Here, the symmetry principles invoked above can be helpful.

(2) The divergence of each diagram was considered separately, missing possible cancellations between different contributions at a given order. This would again demote three-body forces to higher orders than determined by the superficial degree of divergence. It would also require a fine-tuning whose origin would have to be understood. When further resummations of infinitely many diagrams should be necessary beyond LO, naïve dimensional analysis must be amended further. This could happen when the power-counting developed here does not accord to nuclear phenomenology.

(3) Modifications by overlapping divergences should also be explored. Again, they weaken the degree of divergence, but are particularly important for those three-body forces which cannot be rewritten as Nd-interactions.

(4) The problem of "fractional orders": since the LO amplitude involves an infinite number of graphs, equivalent to the solution of a Faddeev equation, the amplitude approaches for large half off-shell momenta generically a power-law behaviour $q^{-s_l(\lambda)-1}$ with irrational and even complex powers. Three-body forces contain therefore fol-

lowing Section 2.5 non-analytic pieces. We assume that this behaviour is changed at higher orders only by integer powers because higher-order corrections involve only a finite number of diagrams after the LO-graphs are summed into the Faddeev equation. At which concrete order a given three-nucleon interaction needs to be included to render observables cut-off independent can therefore become a question beyond the clear prescription according to which only divergences make the inclusion of counter-terms mandatory. A softer criterion is formulated rigorously only with great difficulty.

(5) The two-nucleon propagator (2.1) at the starting point of the derivation was taken to be already renormalised. This should pose no problem as the Faddeev equation was solved without further cut-offs, so that no overlapping divergences occur. Indeed, any "scaleless" regulator (like dimensional regularisation) will lead to the same result.

(6) Partial wave-mixing and -splitting as well as mixing between Wigner-$SU(4)$-symmetric and antisymmetric amplitudes should be considered in more detail. However, we saw already examples where these effects are suppressed. For example, the power-counting in the partial wave with the lowest angular momentum amongst those that mix is unchanged, see the ${}^4S_{3/2}$–${}^4D_{3/2}$–${}^2D_{3/2}$-mixing in Table 3. The higher the partial wave, the closer is its asymptotics to the simplistic expectation (2.17).

(7) We assumed—as usual in EFT—that the typical size of three-nucleon counter-terms is set by the size of their running. There are cases where the finite part of a counter-term is anomalously large and thus should be included already at lower orders than the naïve dimensional estimate suggests. One example is the anomalous isovector magnetic moment of the nucleon which is as large as the inverse expansion parameter of EFT($\not{\pi}$), $\kappa_1 = 2.35 \approx 1/Q$ [41]. Such cases are, however, rare and must be justified with care.

Finally, Blankleider and Gegelia [40] claimed in an unpublished preprint 5 years ago that the ${}^2S_{1/2}$-wave problem can be solved at LO without resorting to a three-body force to stabilise the system against collaps. According to them, if the Faddeev equation has multiple solutions, then only one is equivalent to the series of diagrams drawn in Fig. 1. We focus in this article on the higher partial waves where the Faddeev equation has—as demonstrated in Section 2.3—always unique solutions for integer $l > 0$ and $\lambda \in \{1; -1/2\}$, so that the alleged discrepancy cannot arise. Distracting the reader for a moment, one may, however, point out a few observations which contradict the claim of Ref. [40]. The derivation of the Faddeev equation is just a special case of Schwinger–Dyson equations, which are well known to be derived in the path-integral formalism without resort to perturbative methods, see, e.g., [42, Chapter 10]. One resorts to a "series of diagrams" only for illustrative purposes like in Fig. 1. Recall that in the case of the three-body system, no small expansion parameter exists in which this series can be made to converge absolutely. In addition, and on a less formal level, well-known properties of the three-body system like the Thomas and Efimov effects [14,30] and the Phillips line [6,31,32] are not explained under the assertions of Ref. [40]. These universal properties were recently also tested experimentally, e.g., for particle-loss rates in Bose–Einstein condensates near Feshbach resonances, see [17] for a review.

4. Conclusions and outlook

In this article, the ordering of three-body contributions in three-body systems also coupled to external currents was constructed systematically for any EFT with only contact interactions and an anomalously large two-body scattering length. Evading explicit calculations, the result is based on naïve dimensional analysis [1], improved by the observation that because the problem is non-perturbative already at leading order, the solution to the Faddeev integral equation does for large off-shell momenta not follow a simplistic dimensional estimate, Sections 2.3 and 3.1. This was shown by constructing the analytical solution to the Faddeev equation in that limit for arbitrary angular momentum and spin-parameter. One could thus develop a “partial-wave regularisation” as an alternative to regulate and renormalise the three-body system. A simplistic approach to naïve dimensional analysis fails for systems which are non-perturbative already at leading order.

In order to keep observables insensitive to the details of short-distance physics, one employs the canonical EFT tenet that a three-body force must be included *if and only if* it is needed as counter-term to cancel divergences which cannot be absorbed by renormalising two-nucleon interactions. After determining the superficial degree of divergence of a diagram which contains only two-nucleon interactions in Section 2.2, this was used in Section 2.4 to classify the relative importance of three-body interactions for each channel, also for partial-wave mixing and splitting. With these results, the EFT of three spinless bosons and EFT($\not\pi$) become self-consistent field theories which contain the minimal number of parameters at each order to ensure renormalisability and a manifest power-counting of all forces. Each such three-body counter-term gives rise to one subtraction constant which must be determined by a three-body datum.

It must again be stressed that three-body forces are in EFT($\not\pi$) added not out of phenomenological needs. Rather, they cure the arbitrariness in the short-distance behaviour of the two-body interactions which would otherwise contaminate the on-shell amplitude, and hence make low-energy observables cut-off independent on the level of accuracy of the EFT-calculation. Recall that the theory becomes invalid at short distances as processes beyond the range of validity of EFT($\not\pi$) are resolved, namely the pion-dynamics and quark–gluon substructure of QCD. Three-body forces are thus not introduced to meet data but to guarantee that observables are insensitive to off-shell effects. Only the combination of two-body off-shell and three-body effects is physically meaningful.

Most of the three-nucleon forces in partial waves with angular momentum less than 3 have a *weaker* strength than one would expect from a blind application of naïve dimensional analysis, see Table 3. This might seem an academic disadvantage—to include some higher-order corrections which are not accompanied by new divergences does not improve the accuracy of the calculation; one only appears to have worked harder than necessary. However, it becomes a pivotal point when one hunts after three-body forces in observables: in order to predict the experimental precision necessary to disentangle these effects, the error-estimate of EFT is a crucial tool. For many problems, this makes soon a major difference in the question whether an experiment to determine three-body force effects is feasible at all. One such consequence was discussed in Section 3.2: the $^4S_{3/2}$-wave scattering length is fully determined by two-nucleon scattering observables on the level of ± 0.001 fm according to the power-counting of EFT($\not\pi$) developed here. This is more than a factor of ten more accurate than the present experimental number [37], but supported by the observation that all modern high-precision two-nucleon potentials predict this observable to a similarly high accuracy [38,39]. If a three-nucleon force (even one saturated by pion-exchanges) would occur at the order at which it is blindly expected, the spread in the potential-model predictions should be considerably larger.

The consequences for other observables like the famed A_y-problem [43] should also be explored. This will be particularly simple in EFT($\not\pi$) because the theory is less involved and Table 3 sorts the three-nucleon forces according to their strengths, indicating also their symmetries and the channels in which they contribute on the necessary level of accuracy. The conclusions, conjectures and caveats of Section 3 summarise a number of further interesting directions for future research.

To classify the order at which a given two- or three-nucleon interaction should be added in chiral EFT, the EFT of nuclear physics with pions as explicit degrees of freedom, I suggest to follow a path as for EFT($\not\pi$) which complements the so-far mostly pursued phenomenological approach: at leading order, the theory must be non-perturbative to accommodate the finely tuned real and virtual two-body bound states in the S-waves of two-nucleon scattering. After that, only those local two- and three-nucleon forces are added at each order which are necessary as counter-terms to cancel divergences of the amplitudes at short distances. This mandates a more careful look at the leading-order, non-perturbative scattering amplitudes to determine their ultraviolet-behaviour and superficial degree of divergence, see, e.g., [44] and references therein. It leads at each order and to the prescribed level of accuracy to a cut-off independent theory with the smallest number of experimental input-parameters. The power-counting is thus not constructed by educated guesswork but by rigorous investigations of the renormalisation group properties of couplings and observables by EFT-methods. Work in this direction is under way, see also [45], and the future will show its viability.

Acknowledgements

I am particularly indebted to P.F. Bedaque and U. van Kolck for discussions and encouragement. H.-W. Hammer provided valuable corrections to the manuscript. The warm hospitality and financial support for stays at the INT in Seattle and at the ECT* in Trento was instrumental for this research. In particular, I am grateful to the organisers and participants of the “INT Programme 03-3: Theories of Nuclear Forces and Nuclear Systems”. I also acknowledge discussions with a referee concerning the equivalence of Faddeev equations and series of diagrams, in which, however, no consensus was achieved—as in previous exchanges on the same subject. This work was supported in part by the Bundesministerium für Forschung und Technologie, and by the Deutsche Forschungsgemeinschaft under contracts GR1887/2-2 and 3-1.

Appendix A. Solving the integral equation

A.1. Constructing the solution

The Mellin transformation of a function $f(p)$, see, e.g., [46], is defined as

$$\mathcal{M}[f;s] := \int_0^\infty \mathrm{d}p\, p^{s-1} f(p) \quad \text{if } \int_0^\infty \frac{\mathrm{d}p}{p} \left|f(p)\right|^2 \text{ exists.} \tag{A.1}$$

Applying this to both sides of (2.9) and using the faltung theorem [46, Chapter 4.8], one obtains the algebraic equation

$$\mathcal{M}[t;s] = 8\pi\lambda\mathcal{M}\left[\lim_{k\to 0}\mathcal{K}^{(l)}\left(\frac{3k^2}{4}-\gamma^2;k,p\right);s\right] + \frac{8\lambda}{\sqrt{3}\pi}(-1)^l\mathcal{M}\left[Q_l\left(x+\frac{1}{x}\right);s-1\right]\mathcal{M}[t;s], \tag{A.2}$$

which is easily solved for $\mathcal{M}[t;s]$. Thus, one now only has to apply an inverse Mellin transformation,

$$t(p) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}s\, p^{-s}\mathcal{M}[t;s], \tag{A.3}$$

where the inversion contour must be placed in the strip c in which all of the original Mellin transformations exist.

However, there is no Mellin transform of $\mathcal{K}^{(l)}$ in the limit $\gamma, k \ll p$ because it is proportional to $(k)^l/p^{l+2}$. One therefore has to resort for the inhomogeneous term to a slightly more complicated, "half-plane" transformation [46, Chapter 8.5]:

$$\begin{aligned}
&\mathcal{M}_-\left[\lim_{k\to 0}\mathcal{K}^{(l)}\left(\frac{3k^2}{4}-\gamma^2;k,p\right);s\right] \\
&\quad := \int_0^1 \mathrm{d}p\, p^{s-1}\lim_{k\to 0}\mathcal{K}^{(l)}\left(\frac{3k^2}{4}-\gamma^2;k,p\right) \propto \frac{k^l}{s-l-2}, \\
&\mathcal{M}_+\left[\lim_{k\to 0}\mathcal{K}^{(l)}\left(\frac{3k^2}{4}-\gamma^2;k,p\right);s\right] \\
&\quad := \int_1^\infty \mathrm{d}p\, p^{s-1}\lim_{k\to 0}\mathcal{K}^{(l)}\left(\frac{3k^2}{4}-\gamma^2;k,p\right) \propto -\frac{k^l}{s-l-2}.
\end{aligned} \tag{A.4}$$

These Mellin transforms $\mathcal{M}_-$ and $\mathcal{M}_+$ exist for $\mathrm{Re}[s] > \mathrm{Re}[l+2]$ and $\mathrm{Re}[s] < \mathrm{Re}[l+2]$, respectively. The solution to the integral equation is in this case given by [46, Eq. (8.5.43)]:

$$
\begin{aligned}
t_\lambda^{(l)}(p) = \frac{1}{2\pi \mathrm{i}} \Bigg[& \int\limits_{\sigma_- - \mathrm{i}\infty}^{\sigma_- + \mathrm{i}\infty} \mathrm{d}s\, p^{-s} \frac{8\pi\lambda \mathcal{M}_-\big[\lim_{k\to 0} \mathcal{K}^{(l)}\big(\frac{3k^2}{4} - \gamma^2; k, p\big); s\big]}{1 - \frac{8\lambda}{\sqrt{3}\pi}(-1)^l \mathcal{M}\big[Q_l\big(x + \frac{1}{x}\big); s-1\big]} \\
& + \int\limits_{\sigma_+ - \mathrm{i}\infty}^{\sigma_+ + \mathrm{i}\infty} \mathrm{d}s\, p^{-s} \frac{8\pi\lambda \mathcal{M}_+\big[\lim_{k\to 0} \mathcal{K}^{(l)}\big(\frac{3k^2}{4} - \gamma^2; k, p\big); s\big]}{1 - \frac{8\lambda}{\sqrt{3}\pi}(-1)^l \mathcal{M}\big[Q_l\big(x + \frac{1}{x}\big); s-1\big]} \\
& + \oint \mathrm{d}s\, p^{-s} \frac{S(p)}{1 - \frac{8\lambda}{\sqrt{3}\pi}(-1)^l \mathcal{M}\big[Q_l\big(x + \frac{1}{x}\big); s-1\big]} \Bigg],
\end{aligned}
\tag{A.5}
$$

where $\sigma_- > \mathrm{Re}[l+2]$ and $\sigma_+ < \mathrm{Re}[l+2]$. The denominator is simply the Mellin transform of the resolvent of the Faddeev equation. Not surprisingly, it determines the asymptotics of the solution. The function $S(p)$, determined by the boundary conditions, is in general an analytic function in the strip $\sigma_- < \mathrm{Re}[p] < \sigma_+$ in which the integration contour lies.

We thus see that the particular solution is finally defined everywhere except at those points $\mathrm{Re}[s] = \mathrm{Re}[l] \pm 2$ where $\mathcal{M}_\pm[\lim_{k\to 0} \mathcal{K}^{(l)}]$ does not exist. It is not necessary to perform the contour-integrations leading to an analytic solution here. Rather, we note that

$$
t_\lambda^{(l)}(p) = \sum_{i=1}^{\infty} c_i \frac{k^l}{p^{s_l^{(i)}(\lambda)+1}} \quad \text{with } 1 = \frac{8\lambda}{\sqrt{3}\pi}(-1)^l \mathcal{M}\left[Q_l\left(x + \frac{1}{x}\right); s_l^{(i)}(\lambda)\right], \tag{A.6}
$$

with some fixed coefficients c_i with which the ith zero $s_l^{(i)}(\lambda)$ of the denominator in (A.5) enters at fixed $(l; \lambda)$—unfortunately, there is no closed form for these residues. We used that because $Q_l(x + 1/x)$ is real and symmetric under $x \to 1/x$, the zeroes in the denominator of (A.5) come in quadruplets $\{\pm s_l^{(i)}(\lambda); \pm s_l^{(i)*}(\lambda)\}$. Only the $s^{(i)} := s$ closest to -1 is important for the amplitude at large p, as it provides the strongest UV-dependence. Notice again that only those solutions exist which do not diverge as $p \to \infty$ and for which the Mellin transformation $\mathcal{M}[Q_l(x + \frac{1}{x}); s]$ exists as well.

A.2. How to do an integral

To obtain the zeroes of the denominator—or equivalently Eq. (2.18) in the main text—we now perform the Mellin transformation of $Q_l(x + \frac{1}{x})$. First, one represents the Legendre polynomial by a hyperbolic function [27, Eq. (8.820.2)], and then uses in turn the series-representation for ${}_2F_1$ [27, Eq. (9.100)]:

$$
\begin{aligned}
Q_l\left(x + \frac{1}{x}\right) &= \frac{\sqrt{\pi}\,\Gamma[l+1]}{2^{l+1}\Gamma\big[l + \frac{3}{2}\big]} \left(x + \frac{1}{x}\right)^{-l-1} {}_2F_1\left[\frac{l+2}{2}, \frac{l+1}{2}; l + \frac{3}{2}; \left(x + \frac{1}{x}\right)^{-2}\right] \\
&= \frac{\sqrt{\pi}\,\Gamma[l+1]}{2^{l+1}\Gamma\big[\frac{l}{2}+1\big]\Gamma\big[\frac{l+1}{2}\big]} \\
&\quad \times \sum_{n=0}^{\infty} \frac{\Gamma\big[\frac{l}{2}+1+n\big]\Gamma\big[\frac{l+1}{2}+n\big]}{\Gamma\big[l + \frac{3}{2} + n\big]\Gamma[n+1]} \left(x + \frac{1}{x}\right)^{-(2n+l+1)}.
\end{aligned}
\tag{A.7}
$$

This series is convergent because $(x+\frac{1}{x})^{-2}<1$ for all $x\in[0;\infty]$, cf. [27, Eq. (9.102)]. Now, perform the Mellin transformation of each term using [27, Eq. (3.251.2)]:

$$\int_0^\infty \mathrm{d}x\, x^{2n+s+l}\left(x^2+1\right)^{-(2n+l+1)}=\frac{1}{2}\frac{\Gamma\left[n+\frac{l+s+1}{2}\right]\Gamma\left[n+\frac{l-s+1}{2}\right]}{\Gamma[2n+l+1]}. \tag{A.8}$$

This integral exists for $\mathrm{Re}[2n+l+1]>|\mathrm{Re}[s]|$, and hence for sufficiently large n, i.e., for an infinite, absolutely converging sequence. By analytic continuation, the result can thus be shown to be correct for all n. After a few simple manipulations also with the aid of the doubling formula [27, Eq. (8.335.1)], one can resum the series again:

$$\begin{aligned}
&\mathcal{M}\left[Q_l\left(x+\frac{1}{x}\right);s\right]\\
&\quad=\sqrt{\pi}2^{-(l+2)}\sum_{n=0}^{\infty}4^{-n}\,\frac{\Gamma\left[n+\frac{l+s+1}{2}\right]\Gamma\left[n+\frac{l-s+1}{2}\right]}{\Gamma\left[n+l+\frac{3}{2}\right]\Gamma[n+1]}\\
&\quad=\sqrt{\pi}2^{-(l+2)}\frac{\Gamma\left[\frac{l+s+1}{2}\right]\Gamma\left[\frac{l-s+1}{2}\right]}{\Gamma\left[\frac{2l+3}{2}\right]}\,{}_2F_1\left[\frac{l+s+1}{2},\frac{l-s+1}{2};\frac{2l+3}{2};\frac{1}{4}\right].
\end{aligned} \tag{A.9}$$

Inserting this into (A.6) leads to the algebraic equation (2.18) for the coefficients $s_l(\lambda)$ given in the main text.

When does a solution to the homogeneous version of (2.9) exist? In general, no arbitrary homogeneous terms can be added due to Fredholm's alternative: a non-zero solution exists for a given boundary condition either for the inhomogeneous or for the homogeneous integral equation. This follows also from the considerations leading to (A.5) because the two regions in which

$$\mathcal{M}_-\left[\lim_{k\to 0}\mathcal{K}^{(l)}\left(\frac{3k^2}{4}-\gamma^2;k,p\right);s\right]\quad\text{and}\quad\mathcal{M}_+\left[\lim_{k\to 0}\mathcal{K}^{(l)}\left(\frac{3k^2}{4}-\gamma^2;k,p\right);s\right]$$

are defined do in general not overlap, so that $S(p)$ has no support.

However, when $\mathcal{M}[Q_l(x+\frac{1}{x});s]$ itself exists, then the homogeneous version of the integral equation has a solution. In that case, (A.8) exists for each n, and in particular for $n=0$, so that one must have $\mathrm{Re}[l+1]>|\mathrm{Re}[s]|$. As shown in Section 2.3, the kernel is then singular, circumventing Fredholm's alternative. Danilov [10] discussed the case $(l=0;\ \lambda=1)$, where the Mellin transformation is listed in [27, Eq. (4.296.3)] with the constraint $1>|\mathrm{Re}[s_{l=0}]|$, consistent with our result.

To summarise, all Mellin transformations of the particular solution in (A.5) are well-defined for all (s,l,λ) except for the driving term, and for the back-transformation (A.5). This constrains the values of s to:

$$\mathrm{Re}[s]\neq\mathrm{Re}[l]\pm 2,\qquad \mathrm{Re}[s]>-1. \tag{A.10}$$

The homogeneous part of the Faddeev equation has in general a solution only if the kernel is not compact. This is found for

$$\left|\mathrm{Re}[s]\right|<\mathrm{Re}[l+1]. \tag{A.11}$$

A.3. The full off-shell amplitude

One obtains the solution to the full off-shell Faddeev equation (2.14) easily as follows. Replace

$$8\pi\lambda\mathcal{M}_{\pm}\left[\lim_{k\to 0}\mathcal{K}^{(l)}\left(\frac{3k^2}{4}-\gamma^2;k,p\right);s\right]$$

in Appendix A.1 by

$$\mathcal{M}\left[8\pi\lambda\frac{(-1)^l}{kp}Q_l\left(\frac{p}{k}+\frac{k}{p}\right);s\right]=8\pi\lambda(-1)^l k^{s-2}\mathcal{M}\left[Q_l\left(x+\frac{1}{x}\right);s-1\right]. \quad \text{(A.12)}$$

The asymptotic form given in (2.15) follows now from the analogue to (A.5) and (A.6), keeping in mind that the contours can only be closed in the positive half-plane when $k < p$, and in the negative one when $k > p$. Notice that *both* off-shell momenta must obey the integral equations (2.14).

References

[1] A. Manohar, H. Georgi, Nucl. Phys. B 234 (1984) 189, n.b. Acknowledgement;
H. Georgi, L. Randall, Nucl. Phys. B 276 (1986) 241.
[2] S. Weinberg, Nucl. Phys. B 363 (1991) 3.
[3] U. van Kolck, Prog. Part. Nucl. Phys. 43 (1999) 337, nucl-th/9902015.
[4] S.R. Beane, P.F. Bedaque, W.C. Haxton, D.R. Phillips, M.J. Savage, in: M. Shifman (Ed.), At the Frontier of Particle Physics, World Scientific, Singapore, 2001, nucl-th/0008064.
[5] P.F. Bedaque, U. van Kolck, Annu. Rev. Nucl. Part. Sci. 52 (2002) 339, nucl-th/0203055.
[6] P.F. Bedaque, H.-W. Hammer, U. van Kolck, Nucl. Phys. A 676 (2000) 357, nucl-th/9906032.
[7] H.-W. Hammer, T. Mehen, Phys. Lett. B 516 (2001) 353, nucl-th/0105072.
[8] P.F. Bedaque, G. Rupak, H.W. Grießhammer, H.-W. Hammer, Nucl. Phys. A 714 (2003) 589, nucl-th/0207034.
[9] H. Sadeghi, S. Bayegan, Nucl. Phys. A 753 (2005) 291, nucl-th/0411114.
[10] G.S. Danilov, Sov. Phys. JETP 13 (1961) 349.
[11] P.F. Bedaque, H.-W. Hammer, U. van Kolck, Phys. Rev. Lett. 82 (1999) 463, nucl-th/9809025;
P.F. Bedaque, H.-W. Hammer, U. van Kolck, Nucl. Phys. A 646 (1999) 444, nucl-th/9811046.
[12] K.G. Wilson, Phys. Rev. D 3 (1971) 1818;
S.D. Głazek, K.G. Wilson, Phys. Rev. D 47 (1993) 4657;
S.D. Głazek, K.G. Wilson, Phys. Rev. Lett. 89 (2002) 230401, hep-th/0203088.
[13] E. Braaten, H.W. Hammer, Phys. Rev. Lett. 91 (2003) 102002, nucl-th/0303038.
[14] V. Efimov, Nucl. Phys. A 362 (1981) 45;
V. Efimov, Phys. Rev. C 44 (1991) 2303;
V. Efimov, E.G. Tkachenko, Phys. Lett. B 157 (1985) 108.
[15] I.R. Afnan, D.R. Phillips, Phys. Rev. C 69 (2004) 034010, nucl-th/0312021.
[16] T. Barford, PhD thesis, Manchester University, 2004, nucl-th/0404072;
T. Barford, M.C. Birse, J. Phys. A 38 (2005) 697, nucl-th/0406008.
[17] E. Braaten, H.W. Hammer, cond-mat/0410417.
[18] H.W. Grießhammer, in: U.-G. Meißner, H.-W. Hammer, A. Wirzba (Eds.), Mini-Proceedings of Chiral Dynamics: Theory and Experiment (CD2003), hep-ph/0311212.
[19] H.W. Grießhammer, Nucl. Phys. A 744 (2004) 192, nucl-th/0404073.
[20] In EFT, this was first considered by D.B. Kaplan, Nucl. Phys. B 494 (1997) 471, nucl-th/9610052.
[21] P.F. Bedaque, U. van Kolck, Phys. Lett. B 428 (1998) 221, nucl-th/9710073.

[22] P.F. Bedaque, H.-W. Hammer, U. van Kolck, Phys. Rev. C 58 (1998) R641, nucl-th/9802057.
[23] P.F. Bedaque, H.W. Grießhammer, Nucl. Phys. A 671 (2000) 357, nucl-th/9907077.
[24] J. Schwinger, Hectographed notes on nuclear physics, Harvard University, 1947;
G.F. Chew, M.L. Goldberger, Phys. Rev. 75 (1949) 1637;
F.C. Barker, R.E. Peierls, Phys. Rev. 75 (1949) 3122;
H.A. Bethe, Phys. Rev. 76 (1949) 38.
[25] G.V. Skorniakov, K.A. Ter-Martirosian, Sov. Phys. JETP 4 (1957) 648.
[26] F. Gabbiani, P.F. Bedaque, H.W. Grießhammer, Nucl. Phys. A 675 (2000) 601, nucl-th/9911034.
[27] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, fifth ed., Academic Press, San Diego, 1994.
[28] E. Wigner, Phys. Rev. 51 (1939) 106;
E. Wigner, Phys. Rev. 51 (1939) 947;
E. Wigner, Phys. Rev. 56 (1939) 519.
[29] T. Mehen, I.W. Stewart, M.B. Wise, Phys. Rev. Lett. 83 (1999) 931, hep-ph/9902370.
[30] L.W. Thomas, Phys. Rev. 47 (1935) 903.
[31] A.C. Phillips, G. Barton, Phys. Lett. B 28 (1969) 378.
[32] V. Efimov, E.G. Tkachenko, Few-Body Syst. 4 (1988) 71.
[33] G.P. Lepage, How to renormalize the Schrödinger equation, Lectures given at 9th Jorge Andre Swieca Summer School: Particles and Fields, Sao Paulo, Brazil, 16–28 February, 1997, nucl-th/9706029.
[34] V. Bernard, T.R. Hemmert, U.G. Meißner, Nucl. Phys. A 732 (2004) 149, hep-ph/0307115.
[35] E. Epelbaum, W. Gloeckle, U.G. Meißner, Eur. Phys. J. A 19 (2004) 125, nucl-th/0304037.
[36] T.C. Black, et al., Phys. Rev. Lett. 90 (2003) 192502;
K. Schön, et al., Phys. Rev. C 67 (2003) 044005.
[37] W. Dilg, L. Koester, W. Nistler, Phys. Lett. B 36 (1971) 208.
[38] J.L. Friar, D. Hüber, H. Witała, G.L. Payne, Acta Phys. Polon. B 31 (2000) 749, nucl-th/9908058.
[39] H. Witała, A. Nogga, H. Kamada, W. Glöckle, J. Golak, R. Skibinski, nucl-th/0305028.
[40] B. Blankleider, J. Gegelia, nucl-th/0009007.
[41] H.W. Grießhammer, G. Rupak, Phys. Lett. B 529 (2002) 57, nucl-th/0012096.
[42] C. Itzykson, J.-B. Zuber, Quantum Field Theory, McGraw–Hill, New York, 1980.
[43] See, e.g., D. Hüber, J.L. Friar, Phys. Rev. C 58 (1998) 674, nucl-th/9803038.
[44] S.R. Beane, P.F. Bedaque, M.J. Savage, U. van Kolck, Nucl. Phys. A 700 (2002) 377, nucl-th/0104030.
[45] A. Nogga, in: J. Bijnens, U.-G. Meißner, A. Wirzba (Eds.), Mini-Proceedings of the 337th W.E. Heraeus Seminar: Effective Field Theories in Nuclear, Particle, and Atomic Physics, Bad Honnef, Germany, 15 December, 2004, hep-ph/0502008;
A. Nogga, R.G.E. Timmermans, U. van Kolck, nucl-th/0506005.
[46] P.M. Morse, H. Feshbach, Methods of Theoretical Physics, Part I, McGraw–Hill, New York, 1953.