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How to Classify Three-Body Forces – And Why*

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Abstract. For systems with only short-range forces and shallow two-body bound states, the typical strength of any three-body force in all partial waves, including external currents, is systematically estimated by renormalization-group arguments in the effective field theory of point-like interactions. The underlying principle and some consequences in particular in nuclear physics are discussed. Details and a better bibliography can be found in ref. [1].

1 Introduction

The effective field theory (EFT) of point-like interactions is a model-independent approach to systems without infinite-range forces in atomic, molecular and nuclear physics at very low energies with shallow real or virtual two-body bound states ("dimers"), see, e.g., refs. [2, 3] for reviews. When the size or scattering length a of a two-body system is much larger than the size (or interaction range) R of the constituents, a small, dimension-less parameter $Q = R/a$ allows to classify the typical size of neglected corrections at n-th order beyond leading order (N^{n}LO) as about Q^n. For example, $a \approx 104$ Å and $R \approx 10$ Å in the $^4He_2$ molecule, i.e., $Q \approx \frac{1}{10}$, while $a \approx 4.5$ fm and $R \approx 1.5$ fm in the deuteron, i.e., $Q \approx \frac{1}{3}$ in the "pion-less" EFT, EFT($\not\pi$), where pion exchange between nucleons is not resolved as non-local. Thus, the detailed dynamics on the "high-energy" scale R can vary largely: For example, attractive van-der-Waals forces $\propto 1/r^6$ balance in $^4He_2$ a repulsive core generated by QED; but in nuclear physics, one-pion exchange $\propto 1/r^{[1\cdots3]}$ is balanced by a short-range repulsion whose origin in QCD is not yet understood. It is a pivotal advantage of an EFT that it allows predictions of pre-determined accuracy

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without such detailed understanding – as long as one is interested in low-energy processes, i.e., physics at the scale a, and not R. Even when possible (as in QED – at least at scales ≥ 1 fm), EFTs reduce numerically often highly involved computations of short-distance contributions to low-energy observables by encoding them into a few simple, model-independent constants of contact interactions between the constituents. These in turn can be determined by simpler simulations of the underlying theory, or – when they are as in QCD not (yet) tenable – by fit to data. Universal aspects of few-body systems with shallow bound states are manifest in EFTs, with deviations systematically calculable. In two-body scattering for example, the EFT of point-like interactions reproduces the effective-range expansion, but goes beyond it in the systematic, gauge-invariant inclusion of external currents, relativistic effects, etc.

Take three-body forces (3BFs): They parameterize interactions on scales much smaller than what can be resolved by two-body interactions, i.e., in which 3 particles sit on top of each other in a volume smaller than R^3. Traditionally, they were often introduced *a posteriori* to cure discrepancies between experiment and theory, but such an approach is of course untenable when data are scarce or one- and two-body properties should be extracted from three-body data. But how important are 3BFs in observables? The classification in EFTs rests on the tenet that a 3BF is included if and only if necessary to cancel cut-off dependences in low-energy observables. I outline this philosophy and its results in the following, finding that – independent of the underlying mechanism – 3BFs behave very much alike in such disparate systems as molecular trimers and three-nucleon systems, but do not follow simplistic expectations.

2 Construction

In the Faddeev equation of particle-dimer scattering without 3BFs, Fig. 1, the S-wave two-body scattering amplitude is given by the LO-term of the effective-range expansion. This dimer and the remaining particle "interact" via $\mathcal{K}_l$, the one-particle propagator projected onto relative angular momentum l. Even for small relative on-shell momenta k between dimer and particle, we need the scattering amplitude $t_\lambda^{(l)}(p)$ for *all* off-shell momenta p to determine its value at the on-shell point $p = k$, and hence in particular for p beyond the scale $1/R$ on which a description in terms of point-like constituents is tenable. It is therefore natural to demand that all low-energy observables on a scale $k \sim 1/a$ are insensitive to details of the amplitude at $p \gg 1/R$, namely to form and value of the regulator, form-factor or cut-off chosen. If not, a 3BF must soak up the dependence. In an EFT, this is the

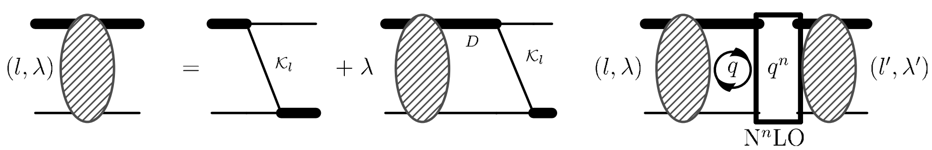

Fig. 1. Left: Integral equation of particle-dimer scattering. **Right**: Generic loop correction (rectangle) at $\mathrm{N}^n\mathrm{LO}$. Thick line (D): Two-body propagator; thin line ($\mathcal{K}_l$): propagator of the exchanged particle; ellipse: LO half off-shell amplitude $t_\lambda^{(l)}(p)$

fundamental tenet: Include a 3BF *if and only if* it is needed as counter-term to cancel divergences which cannot be absorbed by renormalizing two-body interactions. Thus, only combinations of 2- and 3BFs are physically meaningful. With the cut-off variation of the 3BF thus fixed, the initial condition leads to one free parameter fixed from a three-body datum or knowledge of the underlying physics. 3BFs are thus not added out of phenomenological needs but to guarantee that observables are insensitive to off-shell effects.

A Mellin transformation $t_\lambda^{(l)}(p) \propto p^{-s_l(\lambda)-1}$ solves the equation for $p \gg k, 1/a$. The spin-content is then encoded only in the homogeneous term: $\lambda = -\frac{1}{2}$ for 3 nucleons with total spin $\frac{3}{2}$, or for the totally spin and iso-spin (Wigner-)antisymmetric part of the spin-$\frac{1}{2}$ channel; $\lambda = 1$ for 3 identical spin-less bosons and the totally spin and iso-spin (Wigner-)symmetric part of the spin-$\frac{1}{2}$ channel. The asymptotic exponent $s_l(\lambda)$ has to fulfil $\mathrm{Re}[s] > -1$, $\mathrm{Re}[s] \neq \mathrm{Re}[l \pm 2]$, and

$$1 = (-1)^l \, \frac{2^{1-l}\lambda}{\sqrt{3\pi}} \frac{\Gamma\left[\frac{l+s+1}{2}\right]\Gamma\left[\frac{l-s+1}{2}\right]}{\Gamma\left[\frac{2l+3}{2}\right]} \, {}_2F_1\left[\frac{l+s+1}{2}, \frac{l-s+1}{2}; \frac{2l+3}{2}; \frac{1}{4}\right]. \tag{1}$$

This result was first derived in the hyper-spherical approach by Gasaneo and Macek [4][1]. The asymptotics depend thus only but crucially on λ and l. Relevant in the UV-limit are the solutions for which $\mathrm{Re}[s+1]$ is minimal.

At first glance, we would expect the asymptotics to be given by the asymptotics of the inhomogeneous (driving) term:

$$t_\lambda^{(l)}(p) \overset{?}{\propto} \frac{k^l}{p^{l+2}}, \qquad \text{i.e.,} \quad s_l(\lambda) \overset{?}{=} l+1.$$

However, we must sum an infinite number of graphs already at leading order. As Fig. 2 shows, this modifies the asymptotics considerably.

How sensitive are higher-order corrections to the UV-behaviour of $t_\lambda^{(l)}(p)$? The two-body scattering amplitude is systematically improved by including the effective range, higher partial waves, etc. Corrections to three-body observables (including partial-wave mixing) are found by perturbing around the LO solution as in Fig. 1. Most sensitive to unphysically high momenta is each correction at N^{n}LO which is proportional to the n-th power of loop momenta. The question when it becomes cut-off sensitive is now rephrased as: When does the correction diverge as the cut-off is removed, i.e., when is its *superficial degree of divergence* non-negative? The answer by simply counting loop momenta in the diagram is

$$\mathrm{Re}[n - s_l(\lambda) - s_{l'}(\lambda')] \geq 0. \tag{2}$$

We therefore find at which order the first 3BF is needed just by determining when a correction to the three-body amplitude with only two-body interactions becomes dependent on unphysical short-distance behaviour.

[1] My apologies to the authors that I found this reference only after [1] was published

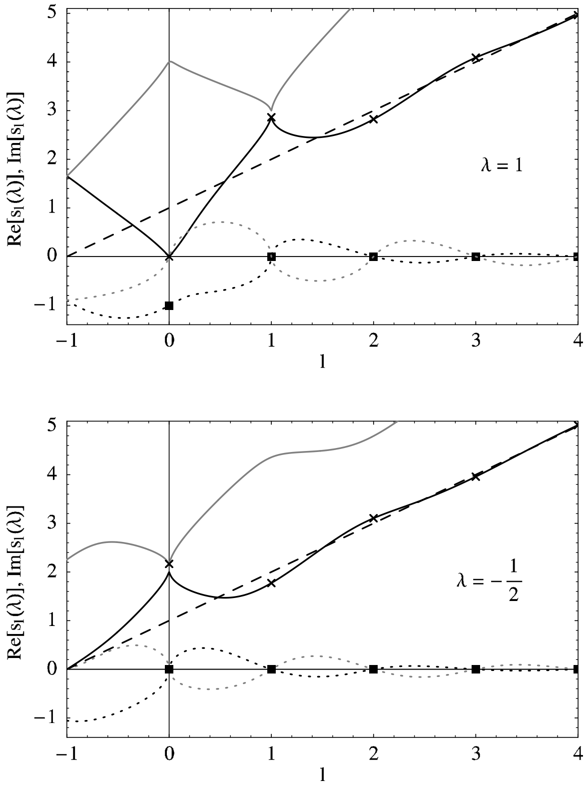

Fig. 2. The first two solutions $s_l(\lambda)$ at $\lambda = 1$ (top) and $\lambda = -\frac{1}{2}$ (bottom). Solid (dotted): real (imaginary) part; dashed: simplistic estimate. Dark/light: first/second solution. Limit cycle and Efimov effect occur only when the solid line lies below the dashed one, and $\mathrm{Im}[s] \neq 0$

It is instructive to re-visit these findings in position space. The Schrödinger equation for the wave function in the hyper-radial dimer-particle distance r,

$$\left[-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{s_l^2(\lambda)}{r^2} - ME\right] F(r) = 0, \tag{3}$$

looks like the one for a free particle with centrifugal barrier. One would thus expect $s_l \overset{?}{=} l + 1$ (hyper-spherical coordinates!). It had, however, already been recognized by Minlos and Faddeev that the centrifugal term is for three bosons ($\lambda = 1$) despite expectations attractive, so that the wave function collapses to the origin and seems infinitely sensitive to very-short-distance physics. In order to stabilize the system against collapse – or, equivalently, remove dependence on details of the cut-off –, a 3BF must be added – or, equivalently, a self-adjoint extension be specified at the origin, i.e., a boundary condition for the wave function must be fixed by a three-body datum. On the other hand, 3BFs are demoted if $s_l > l + 1$: The centrifugal barrier provides *more* repulsion than expected, and hence the wave function is pushed further out, i.e., *less* sensitive to details at distances $r \lesssim R$ where the constituents are resolved as extended. Birse confirmed this recently by a renormalization-group analysis in position space [5].

3 Consequences

About half of the 3BFs for $l \leq 2$ are *weaker*, half *stronger* than one would expect simplistically, see Table 1. The higher partial waves follow this expectation, as the

Table 1. Order of the leading 3BF in particle-dimer scattering, indicating if actual values from Eqs. (1), (2) are stronger ("prom.") or weaker ("dem.") than the simplistic estimate. Last column: Typical size of 3BF in EFT($\not\pi$); in parentheses: size from the simplistic estimate

Partial-wave: *in-out*		Naïve dim. analysis $\mathrm{Re}[s_l(\lambda)+s_{l'}(\lambda')]$	Simplistic $l+l'+2$		Typical size if $Q^n \sim 1/3^n$	
Bosons	Fermions					
S-S	^2S-^2S	LO	N^{2}LO	prom.	100%	(10%)
	^2S-^4D	N$^{3.1}$LO	N^{4}LO	prom.	3%	(1%)
P-P	^2P-^2P	N$^{5.7}$LO	N^{4}LO	dem.	0.2%	(1%)
	^2P-^2P, ^4P-^4P	N$^{3.5}$LO		prom.	2%	
	^2P-^4P	N$^{4.6}$LO		dem.	0.6%	
	^4S-^4S	N$^{4.3+2}$LO	N^{2+2}LO	dem.	0.1%	(1%)
	^4S-^2D	N$^{5.0}$LO	N^{4}LO	dem.	0.4%	
	^4S-^4D	N$^{5.3}$LO		dem.	0.3%	
Higher		$\sim$ as simplistic	N$^{l+l'+2}$LO			

Faddeev equation is then saturated by the Born approximation. The *S*-wave 3BF of spin-less bosons is stronger, while the *P*-wave 3BF is weaker.

That the first *S*-wave 3BF appears already at LO, first described by Bedaque, Hammer and van Kolck, leads to a new renormalization-group phenomenon, the "limit cycle". It explains the Efimov and Thomas effects, and universal correlations, e.g., between particle-dimer scattering length and trimer binding energy (the Phillips line), see the reviews [2, 3]. In general, it appears whenever the kernel of the integral equation is not compact, i.e., $\mathrm{Im}[s] \neq 0$ and $|\mathrm{Re}[s]| < \mathrm{Re}[l+1]$. We finally note that the power counting requires a new, independent 3BF with $2l$ derivatives to enter at N^{2l}LO and provides high-accuracy phase shifts in atom-dimer and nucleon-deuteron scattering, and loss rates close to Feshbach resonances in Bose-Einstein condensates, see, e.g., refs. [2, 3, 6] for details.

Demotion might seem an academic dis-advantage – to include some higher-order corrections which are not accompanied by new divergences does not improve the accuracy of the calculation; one only appears to have worked harder than necessary. However, demotion is pivotal when one wants to predict the experimental precision necessary to dis-entangle 3BFs in observables, and here the error-estimate of EFTs is crucial. In ^{4}He-atom-dimer scattering, where $Q \lesssim \frac{1}{10}$, the experiment cannot yet reveal contributions from 3BFs except in the *S*-wave.

On the other hand, $Q \approx \frac{1}{3}$ in EFT($\not\pi$) of nuclear physics. Now, the demotion or promotion of 3BFs makes all the difference whether an experiment to determine 3BF effects is feasible at all. For example, the quartet-*S*-wave scattering-length in the neutron-deuteron system sets at present the experimental uncertainty in an indirect determination of the doublet scattering length, which in turn is well known to be sensitive to 3BFs. Repeatedly calculated in EFT($\not\pi$),

$$a(^4S_{3/2}) = [5.09(\mathrm{LO}) + 1.32(\mathrm{NLO}) - 0.06(\mathrm{N^2LO})]\ \mathrm{fm} = [6.35 \pm 0.02]\ \mathrm{fm}, \quad (4)$$

(here taken from ref. [6]), it converges nicely at N^{2}LO and agrees very well with experiment, $[6.35 \pm 0.02]$ fm. The theoretical accuracy by neglecting higher-order terms is here estimated conservatively by $Q \approx \frac{1}{3}$ of the difference between the NLO- and $\mathrm{N^2LO}$-result. Table 1 predicts that the first 3BF enters not earlier than $\mathrm{N^6LO}$. Indeed, if the theoretical uncertainty continues to decrease steadily as from NLO to $\mathrm{N^2LO}$, an accuracy of $\pm\left(\frac{1}{3}\right)^3 \times 0.02\,\mathrm{fm} \lesssim \pm 0.001\,\mathrm{fm}$ with input only from two-nucleon scattering can be reached in calculations. This is comparable to the range over which modern high-precision potential-model calculations differ: $[6.344\cdots 6.347]\,\mathrm{fm}$. If the 3BF would occur at $\mathrm{N^4LO}$ as simplistically expected, the error by 3BFs would be $\left(\frac{1}{3}\right)^1 \times 0.02\,\mathrm{fm} \approx 0.007\,\mathrm{fm}$, considerably larger than the spread in the potential-model predictions. Differential cross sections and partial waves are also in excellent agreement with much more elaborate state-of-the-art potential model calculations at energies up to 15 MeV, see, e.g., ref. [6].

The cross section of the triton radiative capture $nd \to t\gamma$ at thermal energies provides another example [7]. Nuclear models give a spread of $[0.49\cdots 0.66]$ mb, depending on the two-nucleon potential, and how the $\Delta(1232)$ as first nucleonic excitation is included. On the other hand, a process at 0.0253 eV [*sic*] incident neutron energy and less than 7 MeV photon energy should be insensitive to details of the deuteron wave function and of a resonance with an excitation energy of 300 MeV. Indeed, the power-counting of 3BFs applies equally with external currents, only that the higher-order interaction in Fig. 1 includes now also the momentum- or energy-transfer from the external source as additional low-energy scales. As no new 3BFs are needed up to $\mathrm{N^2LO}$ to render cut-off independence, the result is completely determined by simple two-body observables:

$$\sigma_{\mathrm{tot}} = [0.485(\mathrm{LO}) + 0.011(\mathrm{NLO}) + 0.007(\mathrm{N^2LO})]\,\mathrm{mb} = [0.503 \pm 0.003]\,\mathrm{mb}. \quad (5)$$

It converges and compares well with the measured value, $[0.509 \pm 0.015]$ mb. The cross section relevant for big-bang nucleo-synthesis ($E_n \approx 0.020\cdots 0.4$ MeV) is also in excellent agreement with data [8].

4 Conclusions

With these findings, the EFT of three-body systems with only contact interactions is a self-consistent, systematic field theory which contains the minimal number of interactions at each order to render the theory renormalizable. Each three-body counter-term gives rise to one subtraction constant which is fixed by a three-body datum. Table 1 sorts the 3BFs by their strengths, their symmetries and the channels in which they contribute at the necessary level of accuracy. Amongst the host of applications in nuclear physics are triton and ^{3}He properties, reactions in big-bang nucleo-synthesis, neutrino astro-physics, the famed nuclear A_y-problem, and the experimental determination of fundamental neutron properties.

The method presented here is applicable to any EFT in which an infinite number of diagrams needs to be summed at LO, e.g., because of shallow bound states. One example is chiral EFT, the EFT of pion-nucleon interactions. Only those local N-body forces are added at each order which are necessary as counter-terms to cancel divergences at short distances. This mandates a careful look at the ultraviolet behaviour of the leading-order, non-perturbative scattering amplitude. It leads at each order and to the prescribed level of accuracy to a cut-off independent theory with the smallest number of experimental input parameters. The power counting is thus not constructed by educated guesswork but by rigorous investigations of the renormalization-group properties of couplings and observables using the methodology of EFT.

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